Tame geometry and extensions of functions – Pawłucki 70

June 23–27, 2025

Booklet of abstracts

Jagiellonian University, 2025

Contents

Abstracts of talks

On arc-analytic geometry

June 27
9:30

Janusz Adamus
University of Western Ontario

Arc-analytic functions and arc-symmetric sets are fairly well understood in the semialgebraic context. Much less is known in the subanalytic category. We will discuss recent progress and some open problems in the quest for a subanalytic arc-analytic Nullstellensatz.

Extensions of k-regulous functions

June 24
17:00

Juliusz Banecki
Jagiellonian University

A k-regulous function is a real-valued function of class C^k defined on a real algebraic set, which outside a set of positive codimension can be written as a quotient of two polynomials. In the talk I will give a survey on the problem of extending k-regulous functions from subvarieties, and a sketch of the recently established solution to the problem in the nonsingular case.

On Pawłucki's Theorem on the Nash Points of Subanalytic Sets

June 25
12:00

André Belotto da Silva
Université Paris Cité

Based on a recently developed rank theorem for local rings presented by Guillaume Rond, we give a new proof of two results of W. Pawłucki:

1. The non-regular locus of a complex or real analytic map is an analytic set.
2. The set of semianalytic or Nash points of a subanalytic set X is a subanalytic set, whose complement has codimension two in X.

Our presentation will include a discussion on the notion of Eisenstein power series and on how the rank theorem can be interpreted in this context.

This is joint work with Octave Curmi and Guillaume Rond.

Pawłucki's contributions to subanalytic geometry

June 23
9:30

Edward Bierstone

University of Toronto

I plan to give an overview of Wiesław Pawłucki's work in subanalytic geometry and its continuing impact, in relation to classical problems on composition and extension of differentiable functions, and developments in o-minimal structures.

Approximation of maps from algebraic polyhedra to real algebraic varieties

July 27
12:00

Marcin Bilski

Jagiellonian University

Given a finite simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ and a real algebraic variety Y, by a $\mathcal{K}$-regular map $|\mathcal{K}| \to Y$ we mean a continuous map whose restriction to every simplex in $\mathcal{K}$ is a regular map. Our result says that if Y is a uniformly retract rational variety and if k, l are integers satisfying $0 \leq l \leq k$, then every $\mathcal{C}^l$ map $|\mathcal{K}| \to Y$ can be approximated in the $\mathcal{C}^l$ topology by $\mathcal{K}$-regular maps of class $\mathcal{C}^k$. One of the tools that we use is the Whitney extension theorem.

This is joint work with W. Kucharz.

Lipschitz Geometry of germs of Real Surfaces

June 24
15:00

Lev Birbrair

Jagiellonian University

I am going to describe old and new results related to Inner, Outer and Ambient Lipschitz geometry of germs of Real semi-algebraic and definable surfaces. The subject is closely related to non-archimedean geometry and Knot Theory. I am going to make an overview. No preliminary knowledge is required.

The Schwartz-MacPherson classes in the framework of Lipschitz stratifications

June 24
16:30

Jean Paul Brasselet

Université d'Aix-Marseille

Schwartz-MacPherson classes were defined in 1965 by Marie-Hélène Schwartz, in the framework of obstruction theory, using "radial" vector fields and Whitney stratifications, then by Robert MacPherson, answering a conjecture by Deligne and Grothendieck. M.-H. Schwartz's definition uses Whitney stratifications and the process is easy to understand. However, it requires delicate and very technical constructions. The use of Lipschitz extension properties and Lipschitz stratifications makes it possible to simplify the construction of M.-H. Schwartz's classes, remaining within the framework of obstruction theory.

This is a joint work with Tadeusz Mostowski (Warsaw, Poland) and Nguyen Thi Bich Thuy (São José do Rio Preto, Brazil).

$\mathcal{C}^m$ extension of semialgebraic functions

June 26
11:00

Jean-Baptiste Campesato

Université d'Angers

We address the question of whether semialgebraic nature on the given data can be preserved by a solution in the Whitney extension problem. More precisely, if a semialgebraic function $f : X \to \mathbb{R}$ defined on a closed subset $X \subset \mathbb{R}^n$ admits a $\mathcal{C}^m$ extension $F : \mathbb{R}^n \to \mathbb{R}$, does it admit a $\mathcal{C}^m$ extension that is also semialgebraic? I will present some positive results for $\mathcal{C}^1$ functions (Aschenbrenner-Thamrongthanyalak), in the planar case (Fefferman-Luli), and for arbitrary n and m involving some loss of regularity.

Work in collaboration with E. Bierstone and P. Milman.

Differentiable approximation of continuous definable maps that preserves the image

June 23
16:30

Antonio Carbone

University of Ferrara

Recently Pawłucki showed that compact sets that are definable in some o-minimal structure admit triangulations of class $\mathcal{C}^p$ for each integer $p \geq 1$. In this talk, we discuss how to use these new techniques of triangulation to show that all continuous definable maps between compact definable sets can be approximated by differentiable maps without changing their image. The result is an interplay between o-minimal geometry and PL geometry and uses a new "surjective definable version" of the finite simplicial approximation theorem.

Linear Equations for C^m Functions

June 23
15:00

Charles Fefferman
Princeton University

The talk presents old results and unsolved problems about equations for unknown C^m functions on $\mathbb{R}^n$.

The simplest such problem is to decide whether a given rational function P/Q is continuous on a given algebraic set V in $\mathbb{R}^n$. Subtleties arise if P and Q have common zeros on V.

More generally the problems concern equations $A_1F_1 + A_2F_2 + \cdots + A_NF_N = f$ where $A_1, \ldots, A_N$ and f are given polynomials and $F_1, \ldots, F_N$ are unknown C^m functions for fixed m.

Joint work with J. Kollár and G.K. Luli.

A converse to Cartan's Theorem B: The extension property for real analytic and Nash sets

June 23
12:00

José F. Fernando
Universidad Complutense de Madrid

In 1957 Cartan proved his celebrated Theorem B and deduced that if $\Omega \subset \mathbb{R}^n$ is an open set and X is a coherent real analytic subset of Ω, then X has the analytic extension property, that is, each real analytic function on X extends to a real analytic function on Ω. So far, the converse implication in its full generality remains unproven. As a matter of fact, in the literature only special cases of non-coherent real analytic sets $X \subset \Omega$ without the extension property appear, some of them due to Cartan himself: mainly real analytic sets $X \subset \Omega$ that have a 'tail', that is, X has a non-pure dimensional irreducible analytic component Y such that the set of points of lower dimension of Y is visible inside X (this means that there exists a point $x \in Y$ such that $\dim(X_x) < \dim(Y)$). This kind of examples generated a general feeling that non-coherent real analytic sets without the analytic extension property may have 'tails'.

The aim of this talk is to show that this is not the case and in fact that the converse to Cartan's Theorem B is true in its full generality:

Theorem 1. *If a subset X of Ω has the analytic extension property, then it is a coherent real analytic subset of Ω.*

Thus, the class of sets with the analytic extension property coincides with the one of coherent real analytic sets. To prove this fact it is crucial the use of Serre's cohomology of sheaves of analytic function germs.

The previous characterization can be extended to the Nash case, which is more sophisticated, because of its finiteness properties and its disappointing behavior with respect to cohomology of sheaves of Nash function germs. However, the information obtained in the analytic case allows us to prove the following:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set and let $X \subset \Omega$ be a set. Each Nash function on X extends to a Nash function on Ω if and only if $X \subset \Omega$ is a coherent Nash set.*

The 'if' implication goes back to some celebrated results of Coste, Ruiz and Shiota, which can be understood as the Nash counterpart to Cartan's Theorem B in the analytic setting. Again, as it happened in the analytic case, the 'only if' implication had been treated only for Nash sets X that have a 'tail'.

Recall that if $M \subset \mathbb{R}^n$ is a Nash manifold, $\mathcal{C}^\infty$ semialgebraic functions on M coincide with Nash functions on M. As an application of our previous strategies, we confront the coherence of a Nash set $X \subset \Omega$ with the fact that each $\mathcal{C}^\infty$ semialgebraic function on X is a Nash function on X. Namely, we prove that these two properties are equivalent for Nash sets. More generally, we provide a full characterization of the semialgebraic sets $S \subset \Omega$ for which $\mathcal{C}^\infty$ semialgebraic functions on S coincide with Nash functions on S.

Joint work with Riccardo Ghiloni.

Outer Lipschitz geometry of surface germs

June 24
11:00

Andrei Gabrielov
Purdue University

A germ at the origin of a closed subset set X of $\mathbb{R}^n$ inherits two metrics from the ambient space: the inner metric, the distance between two points in X defined as the length of a shortest path in X connecting these points, and the outer metric, the distance in $\mathbb{R}^n$ between the two points. A germ is Lipschitz Normally embedded (LNE) if these two metrics are equivalent. Germs X and Y are inner (outer) Lipschitz equivalent if there is a homeomorphism $X \to Y$ bi-Lipschitz with respect to inner (outer) metric. A surface germ is a closed germ of dimension 2 definable in a polynomially bounded o-minimal structure over $\mathbb{R}$, e.g., semialgebraic or subanalytic. Although inner Lipschitz classification of surface germs was done by Lev Birbrair in 1999, outer Lipschitz classification is still an open problem. Even the simplest non-LNE surface germs may exhibit surprisingly complex outer Lipschitz geometry. I am going to review recent progress towards outer Lipschitz classification of definable surface germs, most of it joint work with Lev Birbrair.

Subfield-algebraic geometry and a problem of Wiesław

June 23
11:00

Riccardo Ghiloni
Università di Trento

Let R be a real closed field and let K be any subfield of R. Given a subset X of R^n, we say that $X \subset R^n$ is a K-algebraic set if it is the zero set in R^n of a family of polynomials in the subring $K[\mathtt{x}] := K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ of $R[\mathtt{x}]$. We are interested in studying the algebraic geometry of K-algebraic sets $X \subset R^n$ using only polynomials in $K[\mathtt{x}]$, and comparing it with the usual algebraic geometry of $X \subset R^n$ in which polynomials of the entire ring $R[\mathtt{x}]$ are used. This study generates a new 'hybrid' real algebraic geometry, we call subfield-algebraic geometry, which is particularly rich and interesting in the case K is not a real closed subfield of R, such as $K = \mathbb{Q}$.

In the first part of the talk I will present an overview of the foundational concepts and results of this geometry.

During the Spanish-Polish Mathematical Meeting held in 2023 in Łódź, José F. Fernando and I presented a first draft of the above foundational concepts and results. On that occasion, Wiesław asked whether there exists a notion of stratification for K-algebraic subsets of R^n that is natural in the context of subfield-algebraic geometry and whether each K-algebraic subset of R^n admits such a stratification that is Whitney regular.

In the second part of the talk I will present a complete solution to this problem of Wiesław.

Joint work with José F. Fernando.

Periods and Stokes' Theorem in Tame Geometry

June 26
9:30

Annette Huber-Klawitter
Albert-Ludwigs Universität Freiburg

Periods are complex numbers defined by integrating algebraic differential forms over the rationals over Q-semi-algebraic domains. The set contains many well-known numbers like π, $log(2)$ and the values of the Riemann zeta-function. Their transcendence properties are subject to longstanding deep conjecture going back to Grothendieck. The conjecture hinges on giving cohomological and even motivic interpretations for these numbers. More recently, there have been generalisations to exponential periods where the integrals also involve exponential functions.

In the talk I will discuss a version of Stokes' theorem in tame geometry that is needed in order to make the connection between explicit integrals and cohomology: how is proved using triangulations results of Pawłucki and how it is used.

Metric Zariski Multiplicity Conjecture for multiplicity two

June 24
12:00

Zbigniew Jelonek
IMPAN

We show that if two algebraic $(n-1)$-dimensional cones $P, R \subset \mathbb{C}^n$ with isolated singularities are homeomorphic, then they have the same degree. We also prove that if two algebraic $(n-1)$-dimensional cones $P, R \subset \mathbb{C}^n$ are ambient homeomorphic, then their bases B_P and B_R have the same Euler characteristic. As an application we show that two bi-Lipschitz equivalent Brieskorn-Pham hypersurfaces have the same multiplicities at 0. As the second application we show our main result: if $(X, 0), (Y, 0) \subset (\mathbb{C}^n, 0)$ are germs of analytic hypersurfaces, which are ambient bi-Lipschitz equivalent and $m_0(X) = 2$, then also $m_0(Y) = 2$. At the end of the paper we give also some application of our results to the Arnold-Vassiliew Problem.

Surjectivity of completion for $\mathcal{C}^\infty$ rings. (The algebraic version of Whitney extension theorem)

June 26
16:30

Dmitry Kerner
Ben Gurion University

Consider functions that are $\mathcal{C}^\infty$ at the origin and real-analytic off the origin. The classical lemma of Borel reads (algebraically): the completion homomorphism of this ring at the origin is surjective. Similarly, Whitney extension theorem implies the surjectivity of the completion at closed subsets of $\mathbb{R}^n$. For various applications one needs the surjectivity of such completion maps for more general $\mathcal{C}^\infty$-rings and general filtrations by ideals. We establish the necessary and sufficient conditions for this surjectivity. Moreover, we prove: any element of the completion admits a $\mathcal{C}^\infty$-representative that is real-analytic outside of the locus of completion, has any prescribed vanishing rate "at infinity", and the prescribed positivity behavior at the finite part. Alternatively, the $\mathcal{C}^\infty$-representative can be chosen to satisfy any (compatible) linear conditions.

Jumps of the Milnor numbers in linear deformations

June 24
15:30

Tadeusz Krasiński
University of Łódź

The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor number of f_0 and one of its deformation f_s. In the lecture I will present results on the jump of the Milnor number in linear deformations $f_s = f_0 + sg$ of plane curve singularities f_0.

Joint results with Aleksandra Zakrzewska.

Variants of K-L inequality for definable maps and Thom's a_f condition

June 24
9:30

Krzysztof Kurdyka

Université Savoie Mont Blanc

In the case of definable (in an o-minimal structure) maps with values in $\mathbb{R}^k$, $k > 1$ an analogue of K-L inequality was proven recently. Contrary to the case $k = 1$, in general, a desingularizing function ψ does not extends continously to the boundary of the domain of ψ. We expect that this phenomenon can not happen for maps satisfying Thom's a_f condition.

Ongoing project with M. Bilski and L. Paunescu.

Sharply o-minimal structures

June 26
15:30

Dmitry Novikov

Weizmann Institute of Science

The sharply o-minimal structures are o-minimal structures having an added measure of complexity of the definable sets, akin to the degree of algebraic sets. This complexity should satisfy some natural axioms and provide bounds on the topological complexity of the definable sets, akin to the Bezout theorem for algebraic sets. The sharply o-minimal structures do exist and are helpful in diophantine geometry, particularly in the proof of the Wilkie conjecture. I will survey the known results of this recent theory.

Stratifications and term description over valued fields with analytic structure, uniform Yomdin-Gromov parametrizations

July 27
12:30

Krzysztof Nowak

Jagiellonian University

I establish a certain strong smooth stratification of sets and a term description of functions, which are definable over valued fields (possibly non algebraically closed) with separated analytic structure. The basic tools are: elimination of valued field quantifiers, term structure of definable functions, Lipschitz cell decomposition with preparation of RV-parametrized sets, and a non-Archimedean definable version of Bierstone-Milman's canonical desingularization algorithm, achieved in an earlier paper of mine. As application, I give uniform Yomdin-Gromov parametrizations over Henselian fields K with separated analytic structure, whose leading term structure $RV(K)$ is a group with some finite prime invariant.

Global Complexification of Restricted Log-Exp-Analytic Functions

June 27
11:00

Andre Opris
Universität Passau

In this talk we consider real analytic functions, i.e. functions which can be locally expressed as convergent power series, and ask the following question: Which real analytic functions definable in $\mathbb{R}_{\text{an,exp}}$ admit a holomorphic extension that is also definable in $\mathbb{R}_{\text{an,exp}}$? While finding a holomorphic extension via power series expansion is straightforward, the challenge lies in constructing such an extension in a definably way. The talk is structured as follows: We begin by examining the notion of definable complexification and reviewing some existing results. Although we do not provide a complete answer to the question posed, we introduce a large, non-trivial class of functions definable in $\mathbb{R}_{\text{an,exp}}$. This class includes, for instance, functions which formed by iterated compositions on either side of globally subanalytic functions and the global logarithm. We show that restricted log-exp-analytic functions exhibit analytic properties similar to those of globally subanalytic functions. In particular, we present results concerning their differentiability and definable complexification.

Zariski's dimensionality type of singularities. Case of dimensionality type 2.

June 25
9:30

Adam Parusiński
Université Côte d'Azur

In the 1970s O. Zariski introduced a general equisingularity theory for algebroid and algebraic hypersurfaces over an algebraically closed field of characteristic zero. His theory builds up on understanding the dimensionality type of hypersurface singularities, notion defined recursively by considering the discriminants loci of successive "generic" corank 1 projections. The theory of singularities of dimensionality type 1, that is the ones appearing generically in codimension 1, was developed by Zariski in his foundational papers on equisingular families of plane curve singularities. We give a similar complete description for singularities of dimensionality type 2. Moreover, we show that in this case the generic linear projections are generic in the sense of Zariski, this is still an open problem for the dimensionality type greater than 2.

Joint work with Laurentiu Paunescu, the University of Sydney.

On the semialgebraic Whitney extension problem

June 26
12:00

Armin Rainer
Universität Wien

In 1934, Whitney posed the problem of determining when a function defined on a closed subset of $\mathbb{R}^n$ can be extended to a C^m function on all of $\mathbb{R}^n$. He provided a complete characterization in dimension one. The general case remained open until it was fully resolved by Fefferman in the early 2000s. In this talk, I will discuss a related conjecture: if a semialgebraic function, defined on a closed semialgebraic subset of $\mathbb{R}^n$, extends to a C^m function on $\mathbb{R}^n$, then it also admits a semialgebraic C^m extension. I will present a solution to the $C^{1,1}$ version of this problem by establishing the existence of semialgebraic Lipschitz selections for certain affine-set valued maps. The method extends beyond the semialgebraic category: our results hold in any o-minimal expansion of the real field and apply to $C^{1,\omega}$ functions, where ω is any modulus of continuity definable in the given structure. In this context, I will also discuss the uniform definable extension of definable Whitney jets of class $C^{m,\omega}$.

This is joint work with Adam Parusiński.

On Gabrielov's rank Theorem

June 25
11:00

Guillaume Rond
Université d'Aix-Marseille

This talk is concerned with Gabrielov's rank theorem which is a fundamental result in complex or real local analytic geometry. This result provides conditions to insure that a morphism between rings of convergent power series induces an injective morphism between the associated rings of formal power series. We will present the ideas to prove a generalization of this result, allowing to prove a version of Gabrielov's rank theorem in families. All along the talk we will provide examples bringing to light some pathologies.

This is a joint work with André Belotto and Octave Curmi.

The Nash-Tognoli theorem over $\mathbb{Q}$

June 23
17:00

Enrico Savi
Université Côte d'Azur

In 1973, Tognoli proved that every compact smooth manifold M is diffeomorphic to a nonsingular real algebraic set $M' \subset \mathbb{R}^m$, confirming a conjecture by Nash from 1952. This result, known as the Nash-Tognoli Theorem, led to major advances in real algebraic geometry, particularly regarding the *algebraicity problem*, which seeks to characterize stratified spaces that admit a homeomorphic real algebraic model.

Let K be a subfield of $\mathbb{R}$. We refer to a real K*-algebraic set* as a real algebraic set $X \subset \mathbb{R}^n$ defined by polynomial equations over K. In 2020, Parusiński and Rond proved that every real algebraic set $X \subset \mathbb{R}^n$ is homeomorphic to a real $\overline{\mathbb{Q}}^r$-algebraic set $X' \subset \mathbb{R}^n$, where $\overline{\mathbb{Q}}^r = \overline{\mathbb{Q}} \cap \mathbb{R}$ denotes the real closure of $\mathbb{Q}$. Hence, Parusiński proposed the following open problem:

$\mathbb{Q}$-ALGEBRAICITY PROBLEM: (Parusiński, 2021) Is every algebraic set $X \subset \mathbb{R}^n$ homeomorphic to some $\mathbb{Q}$-algebraic set $X' \subset \mathbb{R}^m$, with $m \geq n$?

Compared to the result of Parusiński and Rond, the fact that $\mathbb{Q}$ is not a real closed field is a crucial difficulty.

The aim of this talk is to present a version over $\mathbb{Q}$ of the Nash-Tognoli theorem, obtained in collaboration with Ghiloni, emphasizing its deep connections to recent advances in algebraic geometry over subfields, as developed by Fernando and Ghiloni. In particular, this result provides a positive answer to the $\mathbb{Q}$-algebraicity problem in the case of compact nonsingular real algebraic sets. Time permitting, I will also discuss other affirmative solutions to the $\mathbb{Q}$-algebraicity problem, as well as some ongoing work.

The talk is mostly based on:

- Riccardo Ghiloni and Enrico Savi, *The topology of real algebraic sets with isolated singularities is determined by the field of rational numbers.* 2023. arXiv: 2302.04142 [math.AG].
- Enrico Savi, *A relative Nash-Tognoli theorem over* $\mathbb{Q}$ *and application to the* $\mathbb{Q}$*-algebraicity problem.* 2024. arXiv: 2302.04673 [math.AG].

Extension of smooth functions from their zero set and the chain rule formula

June 26
15:00

Yosef Yomdin
Weizmann Institute of Science

We consider smooth functions f on the unit ball B^n in $\mathbb{R}^n$, with the max norm $\|f\| = 1$, vanishing on a given subset Z of B^n. The goal is to bound from below higher derivatives of f in terms of the metric geometry of Z. We achieve this goal by combining two main tools:

- Analyzing intersections of Z with "near-straight" smooth curves S, based on a "discrete Crofton formula".
- Analyzing the combinatorics of the chain rule formula, in order to accurately compare the high order derivatives of f and of its restriction to S.
