Subfield-algebraic geometry and a problem of Wiesław

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Tame geometry and extensions of functions - Pawłucki 70

Plan of the talk

1§ Subfield-algebraic geometry: What do we mean?

2§ Foundational concepts and results

3§ A problem of Wiesław on stratifications

Reference

J.F. Fernando, R. Ghiloni: *Subfield-algebraic geometry* (to appear on arXiv)

SUBFIELD-ALGEBRAIC GEOMETRY

JOSÉ F. FERNANDO AND RICCARDO GHILONI

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1§ Subfield-algebraic geometry: What do we mean?

The starting question is:

Question A. *What is the foundational concept of algebraic geometry?*

Complex and real algebraic geometers agree on the answer: algebraic set

(See, for example, page 1 of the books '*Algebraic Geometry I. Complex Projective Varieties*' by Mumford and '*Real Algebraic Geometry*' by Bochnak, Coste and Roy)

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Fix any field L and $n \in \mathbb{N}^*$. Given sets $F \subset L[\mathtt{x}_1, \dots, \mathtt{x}_n]$ and $X \subset L^n$, define

$$\mathcal{Z}_L(F) := \{x \in L^n : f(x) = 0,\ \forall f \in F\},$$

$$\mathcal{I}_L(X) := \{f \in L[\mathtt{x}_1, \dots, \mathtt{x}_n] : f(x) = 0,\ \forall x \in X\}.$$

The set $X \subset L^n$ is *algebraic* if $X = \mathcal{Z}_L(F)$ for some $F \subset L[\mathtt{x}_1, \dots, \mathtt{x}_n]$: this is the standard concept of algebraic set.

Answer A. *In simplest terms, algebraic geometry over the given field* L *is the study of those properties of the algebraic sets* $X \subset L^n$ *that are determined by the ideal* $\mathcal{I}_L(X) \subset L[\mathtt{x}_1, \ldots, \mathtt{x}_n]$.

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Some related fundamental concepts of algebraic geometry over L are:

- Zariski topology of L^n;
- Algebraic dimension $\dim_L(X)$ of algebraic sets $X \subset L^n$;
- Regular and singular points of algebraic sets $X \subset L^n$.

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We are mainly interested in the real case in which L is a real closed field, a *r.c.f.* for short. For example: $L = \mathbb{R}$ and $L = \overline{\mathbb{Q}}^r$ (the field of real algebraic numbers)

To study the real case, we make extensive use of the complex case in which L is an algebraically closed field of characteristic zero, an *a.c.f.* for short. For example: $L = \mathbb{C}$ and $L = \overline{\mathbb{Q}}$ (the field of complex algebraic numbers)

Let $L|K$ be an extension of fields, where L is either an a.c.f. or a r.c.f..

Consider $K[\mathtt{x}_1, \ldots, \mathtt{x}_n] \subset L[\mathtt{x}_1, \ldots, \mathtt{x}_n]$. Given a set $X \subset L^n$, define

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Thus, we have:

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Definition. *We say that the set $X \subset L^n$ is K-*algebraic *if $X = \mathcal{Z}_L(F)$ for some $F \subset K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$, that is,*

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for some $F \subset K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$.

Examples. Set $L|K := \mathbb{R}|\mathbb{Q}$ or $\mathbb{C}|\mathbb{Q}$.

- $\{\sqrt[3]{2}\} = \mathcal{Z}_{\mathbb{R}}(\mathrm{x}_1^3 - 2) \subset \mathbb{R}$ is $\mathbb{Q}$-algebraic, $\{\sqrt[3]{2}\} \subsetneqq \mathcal{Z}_{\mathbb{C}}(\mathrm{x}_1^3 - 2) \subset \mathbb{C}$ is not.

- $\{\sqrt{2}\} \subset \mathbb{R}$ and $\{\sqrt{2}\} \subset \mathbb{C}$ are not $\mathbb{Q}$-algebraic.

- The set $X := \mathcal{Z}_{\mathbb{R}}(\mathrm{x}_1 - \sqrt[3]{2}\mathrm{x}_2) \subset \mathbb{R}^2$ is $\mathbb{Q}$-algebraic, because $X = \mathcal{Z}_{\mathbb{R}}(\mathrm{x}_1^3 - 2\mathrm{x}_2^3)$.

- The set $X_{\mathbb{C}} := \mathcal{Z}_{\mathbb{C}}(\mathrm{x}_1 - \sqrt[3]{2}\mathrm{x}_2) \subset \mathbb{C}^2$ is not, because $X_{\mathbb{C}} \subsetneqq \mathcal{Z}_{\mathbb{C}}(\mathrm{x}_1^3 - 2\mathrm{x}_2^3)$.

- The set $Y := \mathcal{Z}_{\mathbb{R}}(\mathrm{x}_1 + \sqrt{2}\mathrm{x}_2 + \sqrt[4]{2}\mathrm{x}_3) \subset \mathbb{R}^3$ is not $\mathbb{Q}$-algebraic.

Consider again our extension of fields $L|K$, where L is either an *a.c.f.* or a *r.c.f.*.

Question B. *What do we mean by subfield-algebraic geometry, or better by K-algebraic geometry over L, or $L|K$-algebraic geometry for short?*

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Answer B. *$L|K$-algebraic geometry is the study of those properties of the K-algebraic sets $X \subset L^n$ that are determined by the ideal $\mathcal{I}_K(X) \subset K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$.*

Thus, the $L|L$-algebraic geometry is the standard algebraic geometry over L.

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Thus, the $L|L$-algebraic geometry is the standard algebraic geometry over L.

Consistently with the standard $L|L$-case, is it possible to profitably define some basic concepts such as the following?

- K-Zariski topology of L^n;
- K-algebraic dimension $\dim_K(X)$ of K-algebraic sets $X \subset L^n$;
- K-regular and K-singular points of K-algebraic sets $X \subset L^n$.

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Consistently with the standard $L|L$-case, is it possible to profitably define some basic concepts such as the following? YES

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- K-algebraic dimension $\dim_K(X)$ of K-algebraic sets $X \subset L^n$;
- K-regular and K-singular points of K-algebraic sets $X \subset L^n$.

Our main goal is to study the geometry of K-algebraic sets $X \subset L^n$ when L is a *r.c.f.* and K is not. This $L|K$-algebraic geometry is rich in new phenomena.

The main example to keep in mind is $L|K = R|\mathbb{Q}$, where R is an arbitrary *r.c.f.*, e.g., $L|K = \mathbb{R}|\mathbb{Q}$ and $L|K = \overline{\mathbb{Q}}^r|\mathbb{Q}$.

2§ Foundational concepts and results

Let $L|K$ be any extension of fields.

K-Zariski topology of L^n

- $X \subset L^n$ is K-*algebraic* if $X = \mathcal{Z}_L(F)$ for some $F \subset K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$.
- The K-*Zariski topology* $\tau^{L|K}$ *of* L^n is the topology of L^n whose closed sets are the K-algebraic subsets of L^n. $\tau^{L|K}$ is Noetherian.
- A K-algebraic set $X \subset L^n$ is K-*irreducible* if it is irreducible with respect to $\tau^{L|K}$.
- $\tau^{L|K}$ is Noetherian $\Longrightarrow$ every K-algebraic set $X \subset L^n$ has a unique decomposition in irreducible closed subsets, called K-*irreducible components of* $X \subset L^n$.
- Given any $S \subset L^n$, we denote by $\mathrm{Zcl}^K_{L^n}(S)$ the closure of S with respect to $\tau^{L|K}$, called K-*Zariski closure of* S *in* L^n.
  If $L = K$, then $\mathrm{Zcl}^L_{L^n}(S)$ is the usual Zariski closure $\mathrm{Zcl}_{L^n}(S)$ of S in L^n.

- $\mathrm{Zcl}_{L^n}(S) \subset \mathrm{Zcl}^K_{L^n}(S)$ for every $S \subset L^n$. This inclusion can be strict:

$$\mathrm{Zcl}_{\mathbb{R}}(\{\sqrt{2}\}) = \{\sqrt{2}\} \subsetneq \{-\sqrt{2}, \sqrt{2}\} = \mathrm{Zcl}^{\mathbb{Q}}_{\mathbb{R}}(\{\sqrt{2}\}).$$

K-dimension in L^n

- Given any $S \subset L^n$, the *K-dimension* $\dim_K(S)$ *of S (in L^n)* is the Krull dimension of the ring $K[\mathtt{x}_1, \ldots, \mathtt{x}_n]/\mathcal{I}_K(S)$.
- $\dim_L(S) \leqslant \dim_K(S)$ for every $S \subset L^n$.
- Faltings' theorem $\Longrightarrow$ there exist extensions $L|\mathbb{Q}$ and $\mathbb{Q}$-algebraic sets $X \subset L^2$ such that $\dim_L(X) = 0$ and $\dim_{\mathbb{Q}}(X) = 1$.
- (*Subfield-dimension invariance thm*) If L is either an *a.c.f.* or a *r.c.f.* and $X \subset L^n$ is any K-algebraic set, then $\dim_L(X) = \dim_K(X)$.

  In this situation, we simply write $\dim(X) := \dim_L(X) = \dim_K(X)$.

Notation. From now on:

- R is a real closed field,
- K is an ordered subfield of R, endowed with the ordering induced by that of R,
- $C := R[\mathtt{i}]$ is the algebraic closure of R, where $\mathtt{i} := \sqrt{-1}$,
- $\overline{K}^r$ is the algebraic closure of K in R, i.e., the real closure of K.
- $\overline{K}$ is the algebraic closure of K, i.e., $\overline{K} = \overline{K}^r[\mathtt{i}]$,
- $K[\mathtt{x}] := K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ for short.

The main examples: $C|R|K = \mathbb{C}|\mathbb{R}|\mathbb{Q}$ or $\overline{\mathbb{Q}}|\overline{\mathbb{Q}}^r|\mathbb{Q}$

Galois completion and K-bad set

Galois completion. Let G be the Galois group $G(C:K)$.

For each $\psi \in G$, define the isomorphism (of $\mathbb{Q}$-vector spaces) $\psi_n : C^n \to C^n$ and the ring automorphism $\widehat{\psi} : C[\mathtt{x}_1, \ldots, \mathtt{x}_n] \to C[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ by

$$\psi_n(z_1, \ldots, z_n) := (\psi(z_1), \ldots, \psi(z_n)),$$
$$\widehat{\psi}\big(\textstyle\sum_\nu a_\nu \mathtt{x}^\nu\big) := \textstyle\sum_\nu \psi(a_\nu)\mathtt{x}^\nu.$$

Let $Y \subset R^n$ be a $\overline{K}^r$-algebraic set and let $Z := \mathrm{Zcl}_{C^n}(Y)$ be its complexification.

Definition. We define the *(real) Galois completion* T^r *of* $Y \subset R^n$ (w.r.t. $C|K$) by

$$T^r := \textstyle\bigcup_{\psi \in G}(\psi_n(Z) \cap R^n).$$

Galois completion algorithm/theorem

(1) Choose generators $g_1, \ldots, g_r$ of $\mathcal{I}_{\overline{K}^r}(Y)$ in $\overline{K}^r[\mathtt{x}]$, so $Z = \mathcal{Z}_C(g_1, \ldots, g_r)$.

(2) Choose a finite Galois subextension $E|K$ of $\overline{K}|K$ that contains all the coefficients of the polynomials $g_1, \ldots, g_r$ and set $G' := G(E : K)$.

(3) For each $\sigma \in G'$, define $Z^\sigma := \mathcal{Z}_C(g_1^\sigma, \ldots, g_r^\sigma) \subset C^n$, where $g_j^\sigma := \sum_\nu \sigma(a_\nu)\mathtt{x}^\nu$ if $g_j = \sum_\nu a_\nu \mathtt{x}^\nu$.

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(5) $T^r = \mathrm{Zcl}^K_{R^n}(Y)$ and $\dim(Z^\sigma \cap R^n) \leqslant \dim(Y) = \dim(T^r)$ for each $\sigma \in G'$.

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(6) Let $\mathfrak{H} \subset E[\mathtt{x}]$ be the set of all products of the form $\prod_{\sigma \in G'} h_\sigma$, where $h_\sigma \in \{g_1^\sigma, \dots, g_r^\sigma\}$ for each $\sigma \in G'$. For each $h \in \mathfrak{H}$, define

$$P_h(\mathtt{t}) := \prod_{\tau \in G'}(\mathtt{t} - h^\tau) = \mathtt{t}^d + \sum_{j=1}^d (-1)^j q_{hj}\mathtt{t}^{d-j} \in K[\mathtt{x}_1, \dots, \mathtt{x}_n][\mathtt{t}],$$

where d is the order of G'. Set $\mathfrak{G} := \{q_{hj}\}_{h \in \mathfrak{H},\, j \in \{1,\dots,d\}} \subset K[\mathtt{x}_1, \dots, \mathtt{x}_n]$.

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(7) $\mathcal{I}_K(T^r) = \mathcal{I}_K(Y) = \sqrt{\mathfrak{G} K[\mathtt{x}_1, \ldots, \mathtt{x}_n]}$. In particular, $T^r = \mathcal{Z}_R(\mathfrak{G})$.

Example. Let $g := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \sqrt[4]{2}\mathtt{x}_3 \in \overline{\mathbb{Q}}^r[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]$ and let $Y := \mathcal{Z}_{\mathbb{R}}(g) \subset \mathbb{R}^3$. Apply the Galois completion algorithm:

(1) Choose the generator g of $\mathcal{I}_{\overline{\mathbb{Q}}^r}(Y)$ and set $Z := \mathrm{Zcl}_{\mathbb{C}^3}(Y) = \mathcal{Z}_{\mathbb{C}}(g) \subset \mathbb{C}^3$.

(2) Consider the Galois extension $E := \mathbb{Q}(\sqrt[4]{2}, \mathtt{i})|\mathbb{Q}$ and set $G' := G(E : \mathbb{Q}) = D_4$. We have: $G' = \{\sigma_{ab}\}_{a \in \{0,1,2,3\}, b \in \{0,1\}}$, $\sigma_{ab}(\sqrt[4]{2}) = \mathtt{i}^a \sqrt[4]{2}$ and $\sigma_{ab}(\mathtt{i}) = (-1)^b \mathtt{i}$.

(3) Set:

$$
\begin{aligned}
X_0 &:= Z^{\sigma_{00}} \cap \mathbb{R}^3 = Y, \\
X_1 &:= Z^{\sigma_{10}} \cap \mathbb{R}^3 = \{x \in \mathbb{R}^3 : x_1 - \sqrt{2}x_2 = 0, x_3 = 0\}, \\
X_2 &:= Z^{\sigma_{20}} \cap \mathbb{R}^3 = \{x \in \mathbb{R}^3 : x_1 + \sqrt{2}x_2 - \sqrt[4]{2}x_3 = 0\}, \\
X_3 &:= Z^{\sigma_{30}} \cap \mathbb{R}^3 = X_1.
\end{aligned}
$$

(4) Define $X := X_0 \cup X_1 \cup X_2 \cup X_3$

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X_2 &:= Z^{\sigma_{20}} \cap \mathbb{R}^3 = \{x \in \mathbb{R}^3 : x_1 + \sqrt{2}x_2 - \sqrt[4]{2}x_3 = 0\}, \\
X_3 &:= Z^{\sigma_{30}} \cap \mathbb{R}^3 = X_1.
\end{aligned}
$$

(4) Define $X = X_0 \cup X_1 \cup X_2$

Example. Let $g := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \sqrt[4]{2}\mathtt{x}_3 \in \overline{\mathbb{Q}}^r[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]$ and let $Y := \mathcal{Z}_{\mathbb{R}}(g) \subset \mathbb{R}^3$. Apply the Galois completion algorithm:

(1) Choose the generator g of $\mathcal{I}_{\overline{\mathbb{Q}}^r}(Y)$ and set $Z := \mathrm{Zcl}_{\mathbb{C}^3}(Y) = \mathcal{Z}_{\mathbb{C}}(g) \subset \mathbb{C}^3$.

(2) Consider the Galois extension $E := \mathbb{Q}(\sqrt[4]{2}, \mathtt{i})|\mathbb{Q}$ and set $G' := G(E : \mathbb{Q}) = D_4$. We have: $G' = \{\sigma_{ab}\}_{a \in \{0,1,2,3\}, b \in \{0,1\}}$, $\sigma_{ab}(\sqrt[4]{2}) = \mathtt{i}^a \sqrt[4]{2}$ and $\sigma_{ab}(\mathtt{i}) = (-1)^b \mathtt{i}$.

(3) Set:

$$\begin{aligned} X_0 &:= Z^{\sigma_{00}} \cap \mathbb{R}^3 = Y, \\ X_1 &:= Z^{\sigma_{10}} \cap \mathbb{R}^3 = \{x \in \mathbb{R}^3 : x_1 - \sqrt{2}x_2 = 0, x_3 = 0\}, \\ X_2 &:= Z^{\sigma_{20}} \cap \mathbb{R}^3 = \{x \in \mathbb{R}^3 : x_1 + \sqrt{2}x_2 - \sqrt[4]{2}x_3 = 0\}, \\ X_3 &:= Z^{\sigma_{30}} \cap \mathbb{R}^3 = X_1. \end{aligned}$$

(4) Define $X = X_0 \cup X_1 \cup X_2$. Thus, $T^r := X$ is the Galois completion of $Y \subset \mathbb{R}^3$.

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\end{aligned}
$$

(4) Define $X = X_0 \cup X_1 \cup X_2$. Thus, $T^r := X$ is the Galois completion of $Y \subset \mathbb{R}^3$.

(5) $X = \operatorname{Zcl}^{\mathbb{Q}}_{\mathbb{R}^3}(Y)$ is a $\mathbb{Q}$-irreducible $\mathbb{Q}$-algebraic set of dimension 2 such that $\mathcal{I}_{\mathbb{R}}(X) = (\prod_{a=0}^{3} g^{\sigma_{a0}}) = (\mathtt{x}_1^4 - 4\mathtt{x}_1^2\mathtt{x}_2^2 + 8\mathtt{x}_1\mathtt{x}_2\mathtt{x}_3^2 + 4\mathtt{x}_2^4 - 2\mathtt{x}_3^4)$

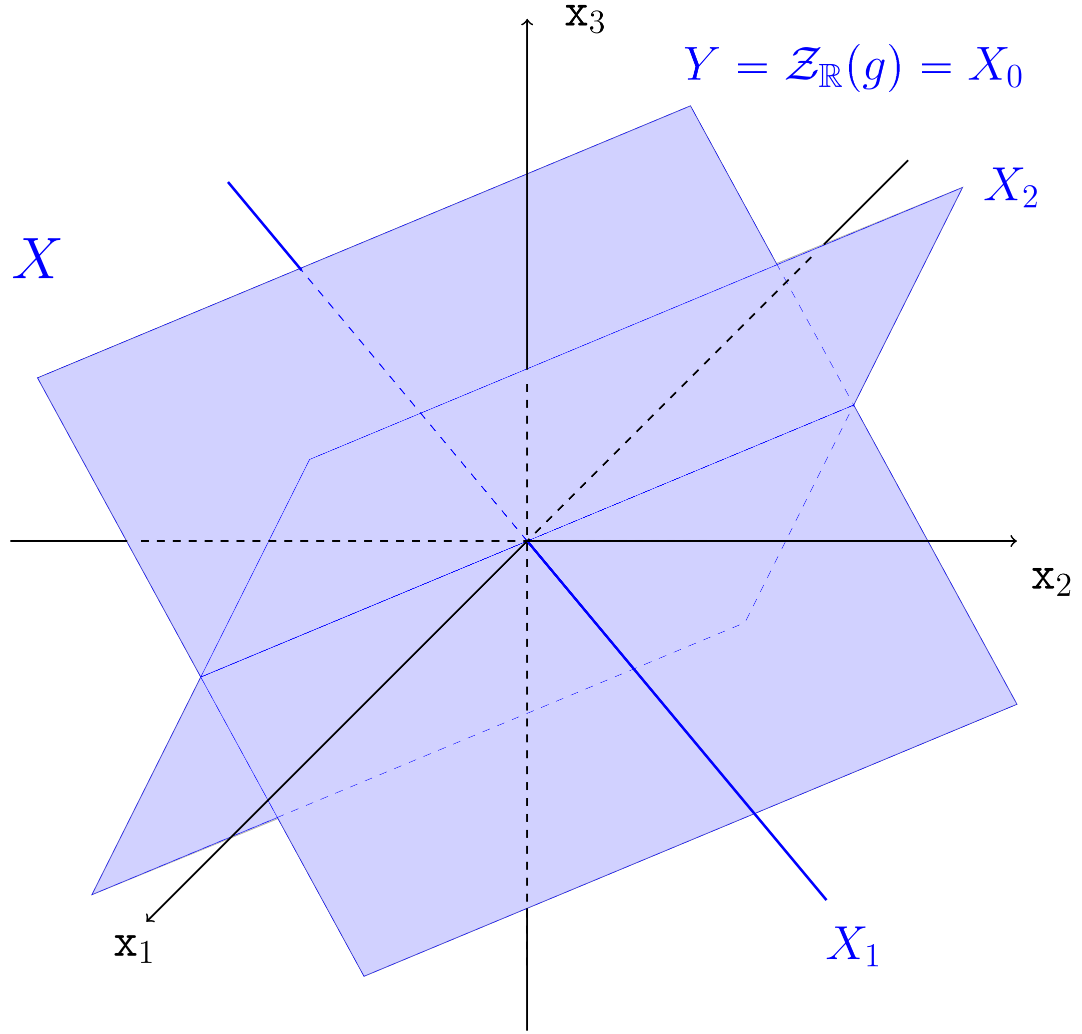

Figure 1: The Galois completion X of Y and its three $\mathbb{R}$-irreducible components: the planes X_0, X_2 and the line X_1 of $\mathbb{R}^3$. The set $X \subset \mathbb{R}^3$ coincides with the $\mathbb{Q}$-Zariski closure of Y and is $\mathbb{Q}$-irreducible.

Example. Let $g := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \sqrt[4]{2}\mathtt{x}_3$ and $p := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \mathtt{x}_3$ in $\overline{\mathbb{Q}}^r[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]$. Define $W := \mathcal{Z}_{\mathbb{R}}(p)$ and apply the Galois completion algorithm to $Y \cup W = \mathcal{Z}_{\mathbb{R}}(gp)$:

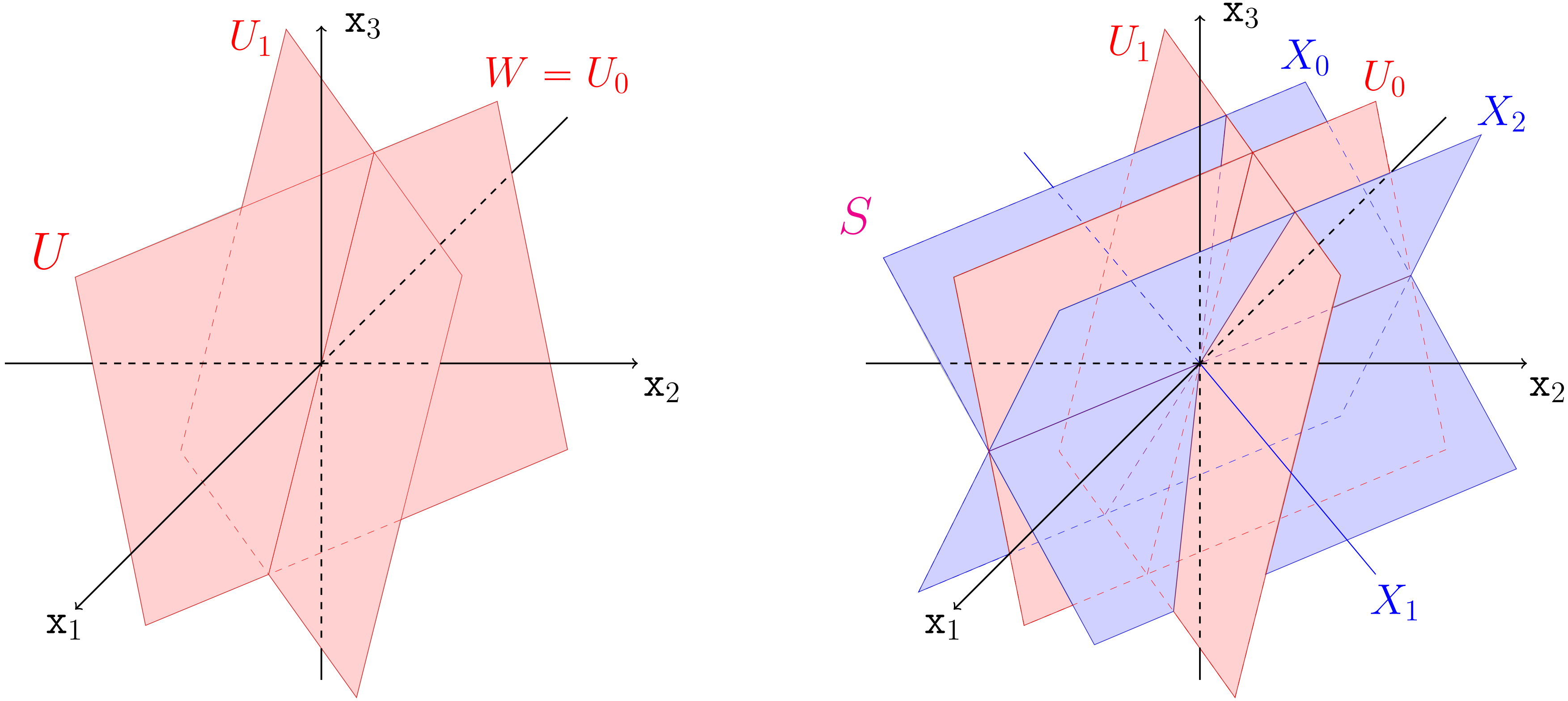

Figure 2: The Galois completion U of $W = U_0$ on the left and the Galois completion S of $Y \cup W = X_0 \cup U_0$ on the right. The set $S \subset \mathbb{R}^3$ is $\mathbb{Q}$-algebraic, has $X = X_0 \cup X_1 \cup X_2$ and $U = U_0 \cup U_1$ as $\mathbb{Q}$-irreducible components and the four planes X_0, X_2, U_0 and U_1 as $\mathbb{R}$-irreducible components.

K-**bad set.** Let $X \subset R^n$ be a K-algebraic set of dimension d.

Suppose $X \subset R^n$ is K-irreducible. Choose a $\overline{K}^r$-irreducible component Y of X of dimension d and define $Z := \mathrm{Zcl}_{C^n}(Y) \subset C^n$. We have:

$$T^r = \bigcup_{\sigma \in G'} (Z^\sigma \cap R^n) = X.$$

Let $G'^\bullet$ be the subset of G' of all σ such that $\dim(Z^\sigma \cap R^n) < d$. We define the K-*bad set* $B_K(X)$ *of* X as follows:

$$B_K(X) := \bigcup_{\sigma \in G'^\bullet} (Z^\sigma \cap R^n).$$

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Remark. In this K-irreducible case, we can prove that $\{Z^\sigma\}_{\sigma \in G'}$ is the family (with possible repetitions) of all the C-irreducible components of $\mathrm{Zcl}^K_{C^n}(X) \subset C^n$.

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$$B_K(X) := \bigcup_{\sigma \in G'^\bullet} (Z^\sigma \cap R^n).$$

Suppose $X \subset R^n$ is K-reducible. Let $X_1, \ldots, X_s$ be the K-irreducible components of X and let I be the set of indices $i \in \{1, \ldots, s\}$ such that $\dim(X_i) = d$. We define the *K-bad set $B_K(X)$ of X* as follows:

$$B_K(X) := \bigcup_{i \in I} B_K(X_i) \cup \bigcup_{i \in \{1,\ldots,s\} \setminus I} X_i.$$

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$$B_K(X) := \bigcup_{i \in I} B_K(X_i) \cup \bigcup_{i \in \{1,\ldots,s\} \setminus I} X_i.$$

Remark. $B_K(X)$ is always empty if K is a real closed field.

Example.

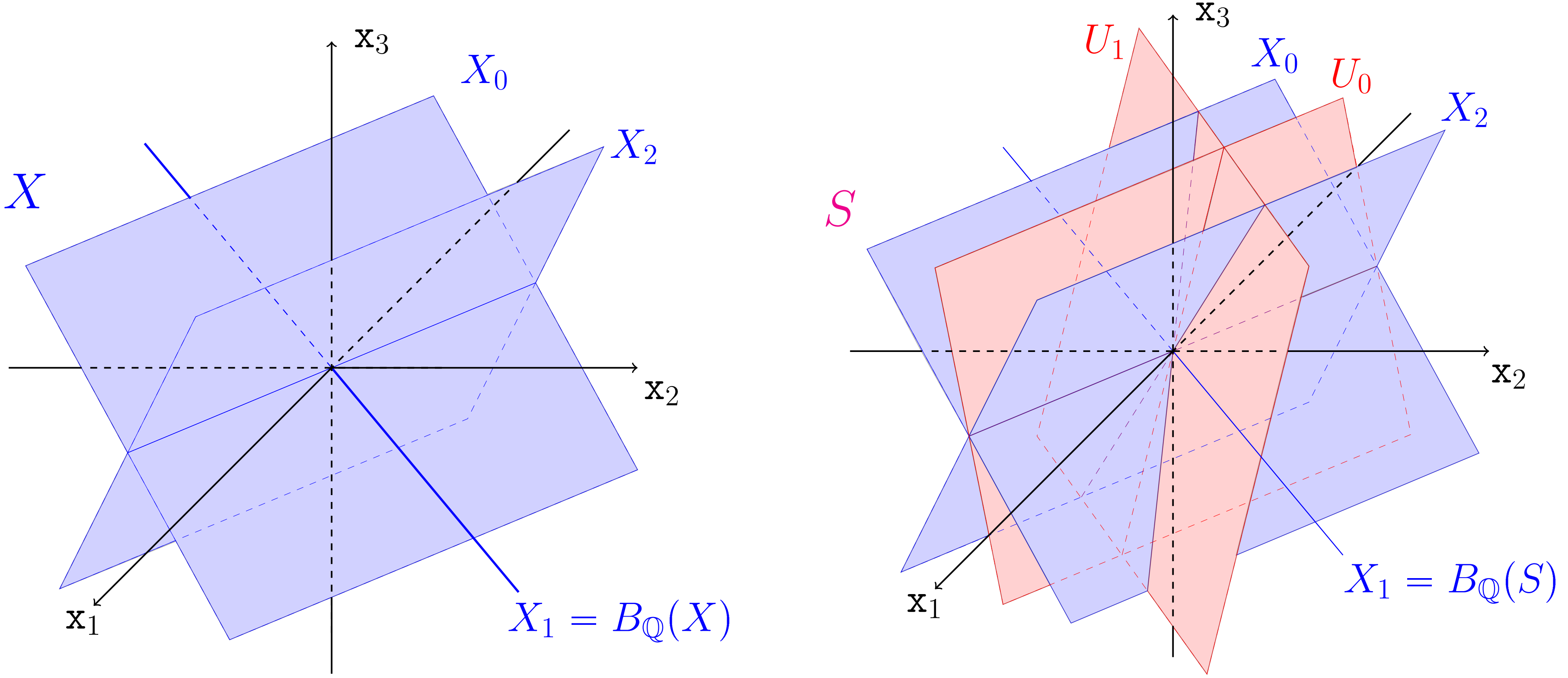

Figure 3: $B_{\mathbb{Q}}(X) = B_{\mathbb{Q}}(S) = X_1$

Example. If $X := \mathcal{Z}_{\mathbb{R}}(\mathtt{x}_1 - \sqrt[3]{2}\mathtt{x}_2) = \mathcal{Z}_{\mathbb{R}}(\mathtt{x}_1^3 - 2\mathtt{x}_2^3) \subset \mathbb{R}^2$, then $B_{\mathbb{Q}}(X) = \{(0,0)\}$.

Regular and singular points. For short, define $R[\mathtt{x}] := R[\mathtt{x}_1, \dots, \mathtt{x}_n]$.

Let $X \subset R^n$ be a K-algebraic set of dimension d, let $a = (a_1, \dots, a_n) \in X$ and let $\mathfrak{n}_a$ be the maximal ideal $(\mathtt{x}_1 - a_1, \dots, \mathtt{x}_n - a_n)$ of $R[\mathtt{x}]$.

Definition. We define the K-*local ring* $\mathcal{R}^K_{X,a}$ *of* X *at* a as

$$\mathcal{R}^K_{X,a} := R[\mathtt{x}]_{\mathfrak{n}_a} / (\mathcal{I}_K(X) R[\mathtt{x}]_{\mathfrak{n}_a}).$$

Remark. $\mathcal{R}^R_{X,a}$ is the usual local ring $\mathcal{R}_{X,a}$ of the algebraic set $X \subset R^n$.

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Definition. a is a K-*regular point of* X if $\mathcal{R}^K_{X,a}$ is a regular local ring of dimension d. If not, a is said to be a K-*singular point of* X. We denote by $\mathrm{Reg}^K(X)$ the set of all K-regular points of X and $\mathrm{Sing}^K(X)$ the set of all K-singular points of X.

Remark. $\mathrm{Reg}^R(X)$ is the usual regular locus $\mathrm{Reg}(X)$ and $\mathrm{Sing}^R(X)$ is the usual singular locus $\mathrm{Sing}(X)$ of the algebraic set $X \subset R^n$.

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Theorem (Jacobian criterion)**.** Let $X \subset R^n$ be a K-algebraic set, let $d := \dim(X)$ and let $a \in X$. Then a is a K-regular point of X if and only if there exist $f_1, \ldots, f_{n-d} \in \mathcal{I}_K(X)$ and an Euclidean open neighborhood U of a in R^n such that

$$\mathrm{rk}\Big(\frac{\partial f_i}{\partial \mathtt{x}_j}(a)\Big)_{i=1,\ldots,n-d,\, j=1,\ldots,n} = n-d \text{ and } X \cap U = \mathcal{Z}_R(f_1, \ldots, f_{n-d}) \cap U.$$

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Theorem (Structure theorem)**.** Let $X \subset R^n$ be a K-algebraic set of dimension d:

(1) $a \in \mathrm{Reg}^K(X)$ if and only if a belongs to a unique K-irreducible component X' of X of dimension d and $a \in \mathrm{Reg}^K(X')$.

(2) $\mathrm{Sing}^K(X)$ is a K-algebraic subset of R^n of dimension $< d$. Thus, $\mathrm{Reg}^K(X)$ is a proper K-Zariski open subset of X.

(3) If X is K-irreducible and $\{g_1, \ldots, g_s\}$ is a system of generators of $\mathcal{I}_K(X)$ in $K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$, then

$$\mathrm{Sing}^K(X) = \Big\{a \in X : \; \mathrm{rk}\Big(\frac{\partial g_i}{\partial \mathtt{x}_j}(a)\Big)_{i=1,\ldots,s,\, j=1\ldots,n} < n-d\Big\}.$$

Theorem (Comparison theorem)**.** Let $X \subset R^n$ be a K-algebraic set of dimension d. We have:

(1) $\mathrm{Sing}(X)$ and $B_K(X)$ are $\overline{K}^r$-algebraic subsets of R^n of dimension $< d$, and

$$\mathrm{Sing}^K(X) = \mathrm{Sing}(X) \cup B_K(X).$$

(2) $\mathrm{Reg}(X)$ is a non-empty $\overline{K}^r$-Zariski open subset of X and

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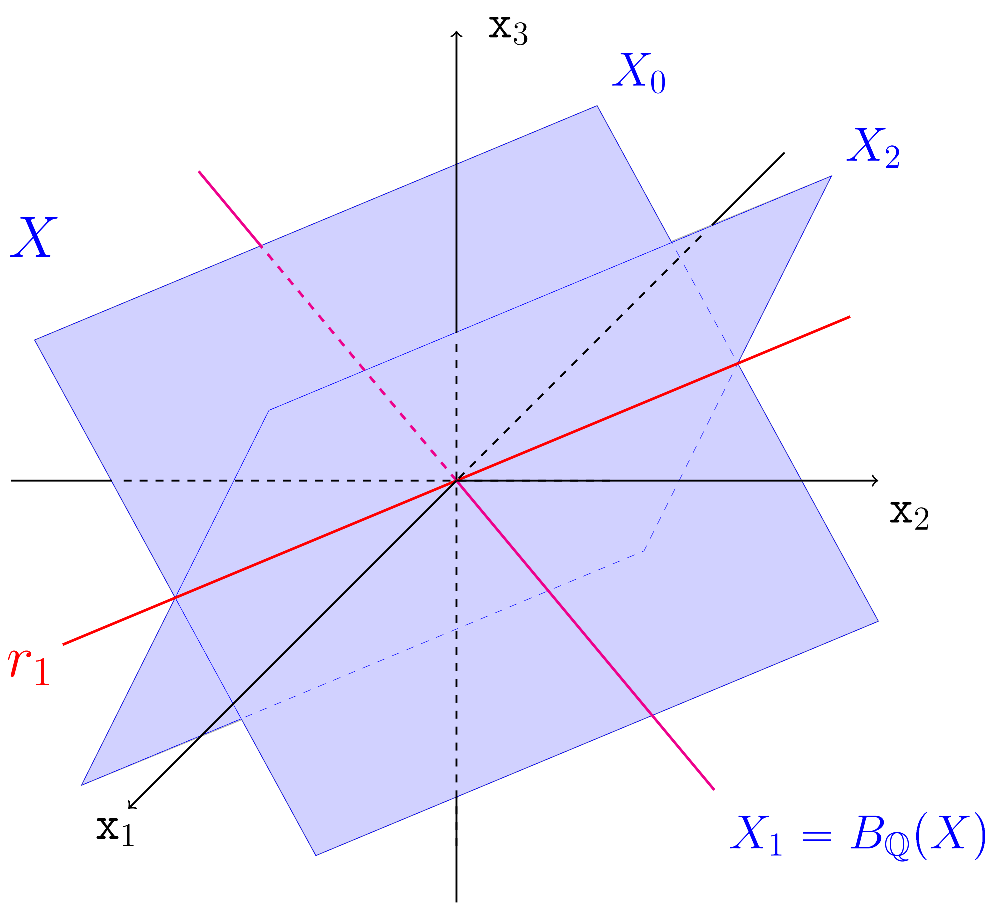
Example. If $X = \mathcal{Z}_{\mathbb{R}}(\mathtt{x}_1 - \sqrt[3]{2}\mathtt{x}_2) = \mathcal{Z}_{\mathbb{R}}(\mathtt{x}_1^3 - 2\mathtt{x}_2^3) \subset \mathbb{R}^2$, then

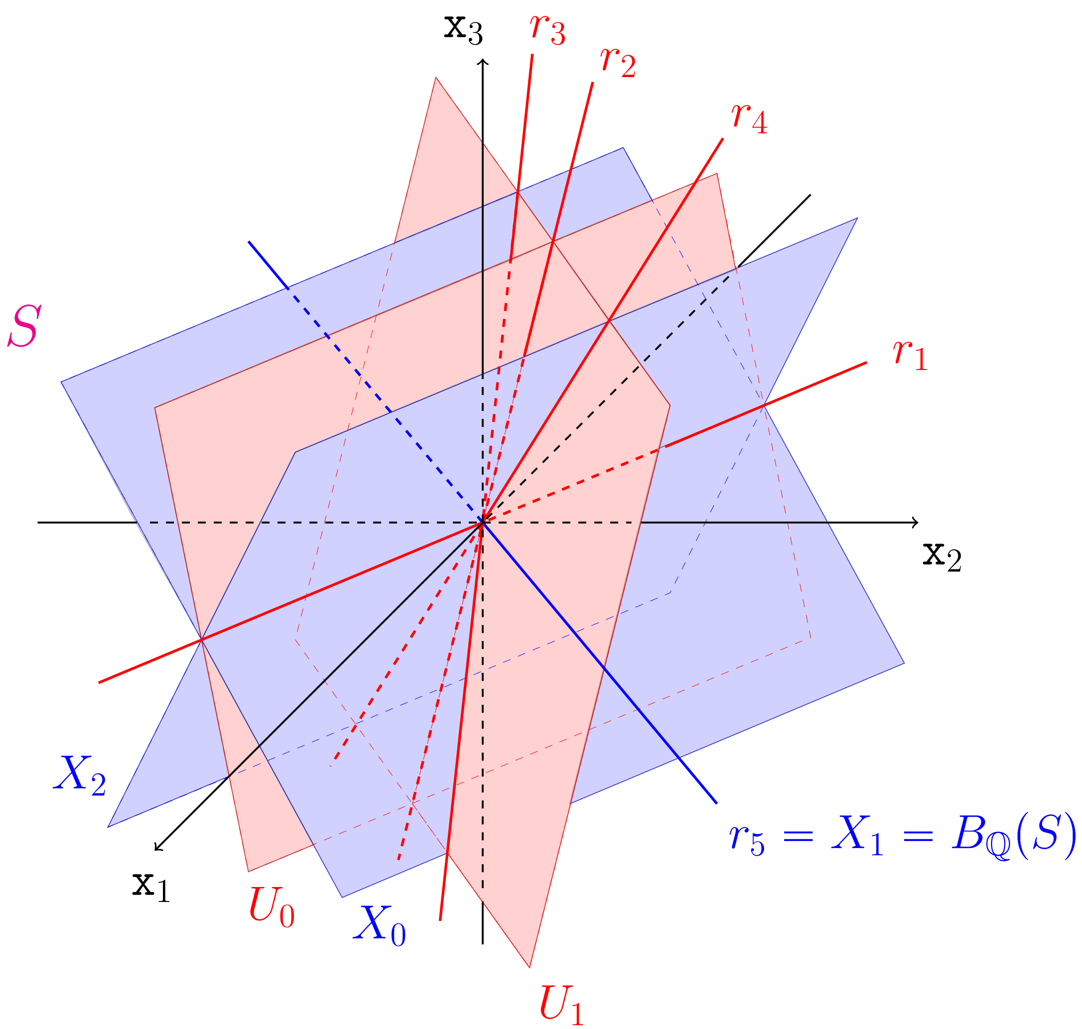
$$\operatorname{Sing}(X) = \varnothing, \qquad B_{\mathbb{Q}}(X) = \{(0,0)\}, \qquad \operatorname{Sing}^{\mathbb{Q}}(X) = \{(0,0)\}.$$

Example. If $g := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \sqrt[4]{2}\mathtt{x}_3$, $X_0 := \mathcal{Z}_\mathbb{R}(g) \subset \mathbb{R}^3$ and $X := \mathrm{Zcl}^{\mathbb{Q}}_{\mathbb{R}^3}(X_0)$, then



$$\mathrm{Sing}(X) = X_1 \cup r_1, \qquad B_\mathbb{Q}(X) = X_1, \qquad \mathrm{Sing}^{\mathbb{Q}}(X) = X_1 \cup r_1.$$

Example. If $p := \mathtt{x}_1 + \sqrt{2}\mathtt{x}_2 + \mathtt{x}_3$, $U_0 := \mathcal{Z}_{\mathbb{R}}(p) \subset \mathbb{R}^3$ and $S := \operatorname{Zcl}^{\mathbb{Q}}_{\mathbb{R}^3}(X_0 \cup U_0)$:



$$\operatorname{Sing}(S) = r_1 \cup r_2 \cup r_3 \cup r_4, \qquad B_{\mathbb{Q}}(S) = r_5, \qquad \operatorname{Sing}^{\mathbb{Q}}(S) = r_1 \cup r_2 \cup r_3 \cup r_4 \cup r_5$$

3§ A problem of Wiesław on stratifications

Recall that R is a real closed field and K is an ordered subfield of R, e.g., $R|K = \mathbb{R}|\mathbb{Q}$.

During the Spanish-Polish Mathematical Meeting held in 2023 in Łódź, we presented a first draft of this paper. On that occasion, Wiesław formulated the following problem consisting of two questions:

(1) *Is there a notion of stratification for K-algebraic subsets of R^n that is natural in the context of $R|K$-algebraic geometry?*

(2) *Is it true that every K-algebraic subset of R^n admits such a stratification that is Whitney regular?*

I present below an affirmative solution to this problem.

Definition. Let S be a subset of R^n.

- We say that S is a K-*semialgebraic set* if $S = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in R^n : f_{ij} *_{ij} 0\}$ for some $s, r_1, \ldots, r_s \in \mathbb{N}^*$, $f_{ij} \in K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ and $*_{ij} \in \{>, =\}$.

- Suppose S is K-semialgebraic. We say that S is K-*semialgebraically connected* if there do not exist two K-semialgebraic sets $C_1 \subset R^n$ and $C_2 \subset R^n$ such that C_1 and C_2 are disjoint proper closed subsets of S and $C_1 \cup C_2 = S$.

Remark. Let $S \subset R^n$ be a K-semialgebraic set. The semialgebraically connected components of S, viewed as a usual semialgebraic set, are K-semialgebraic in R^n. In particular, S is K-semialgebraically connected if and only if it is semialgebraically connected. This was proved by Heintz, Roy and Solernó in 1994.

Definition. (à la Whitney) Let M be a subset of R^n. We say that $M \subset R^n$ is a *K-algebraic partial manifold of dimension* m if M is K-semialgebraic and either M is open in R^n or there exists $m \in \{0, \ldots, n-1\}$ with the following property: for every $p \in M$, there exist $f_1, \ldots, f_{n-m} \in \mathcal{I}_K(M)$ and an open neighborhood U of p in R^n such that the gradients $\nabla f_1(p), \ldots, \nabla f_{n-m}(p)$ are linearly independent in R^n and $M \cap U = \mathcal{Z}_R(f_1, \ldots, f_{n-m}) \cap U$.

Remark. Let $M \subset R^n$ be a K-semialgebraic set and let $M_K := \mathrm{Zcl}^K_{R^n}(M)$. The following assertions are equivalent:

- $M \subset R^n$ is a K-algebraic partial manifold.
- M is an open subset of the K-nonsingular locus $\mathrm{Reg}^K(M_K)$ of M_K.
- There exists a K-algebraic set $X \subset R^n$ such that M is an open subset of $\mathrm{Reg}^K(X)$.

Remark. Every K-algebraic set $X \subset R^n$ can be written as a finite disjoint union of K-algebraic partial submanifolds of R^n. The reason is that $\operatorname{Sing}^K(X)$ is K-algebraic in R^n and of dimension $< \dim(X)$.

Definition. Let $S \subset R^n$ be a K-semialgebraic set. A *K-algebraic stratification of S* is a finite partition $\{M_i\}_{i \in I}$ of S with the following two properties:

(i) Each $M_i \subset R^n$ is a K-semialgebraically connected K-algebraic partial manifold.

(ii) If $M_j \cap \mathrm{Cl}_{R^n}(M_i) \neq \varnothing$ for some $i, j \in I$ with $i \neq j$, then $M_j \subset \mathrm{Cl}_{R^n}(M_i)$.

Remark. We say that a set $M \subset R^n$ is a *K-Nash manifold* if it is both a Nash manifold (in the usual semialgebraic sense) and a K-semialgebraic set.

Evidently, a K-algebraic partial manifold is a K-Nash manifold. The converse is false, e.g., $M := \{y^3 - x^3(1 + x^2) = 0\} \subset R^2$.

An R-algebraic stratification of a semialgebraic set $S \subset R^n$ is a usual semialgebraic stratification of S (with Nash strata). The converse is true up to refinements.

Theorem. *Every K-semialgebraic set $S \subset R^n$ admits a Whitney regular K-algebraic stratification $\{M_i\}_{i\in I}$.*

In addition if $\{S_\lambda\}_{\lambda\in\Lambda}$ is any finite family of K-semialgebraic subsets of R^n contained in S then we can assume that $\{M_i\}_{i\in I}$ is compatible with $\{S_\lambda\}_{\lambda\in\Lambda}$.

Examples. Let $R|K = \mathbb{R}|\mathbb{Q}$.

(1) Let X be the $\mathbb{Q}$-algebraic line $\{x-\sqrt[3]{2}y = 0\} = \{x^3-2y^3 = 0\}$ of $\mathbb{R}^2$. Then $\{X\}$ is a Whitney regular $\mathbb{R}$-algebraic stratification, but it is not a $\mathbb{Q}$-algebraic stratification because $\mathrm{Sing}^{\mathbb{Q}}(X) = \{(0,0)\}$. If $X_\pm := X \cap \{\pm x > 0\}$ and $X_0 := \{(0,0)\}$, then $\{X_+, X_-, X_0\}$ is a Whitney regular $\mathbb{Q}$-algebraic stratification.

(2) Let W be the $\mathbb{Q}$-algebraic Whitney umbrella defined by

$$W := \{y^2 - \sqrt[3]{2} z x^2 = 0\} = \{y^6 - 2z^3 x^6 = 0\} \subset \mathbb{R}^3.$$

Define $W_\pm := W \cap \{\pm x > 0\}$, $Z := W \cap \{x = 0\} = \{x = y = 0\} = \mathrm{Sing}(W)$, $Z_\pm := Z \cap \{\pm z > 0\}$ and $Z_0 := \{(0,0,0)\}$. The partition $\{W_+, W_-, Z_+, Z_-, Z_0\}$ is a Whitney regular $\mathbb{R}$-algebraic stratification of W, but it is not a $\mathbb{Q}$-algebraic stratification:

$$B_\mathbb{Q}(W) = \{y^2 = zx^2 = 0\} = Z \cup X, \text{ where } X := \{y = z = 0\}, \text{ so}$$

$$\mathrm{Sing}^\mathbb{Q}(W) = \mathrm{Sing}(W) \cup B_\mathbb{Q}(W) = Z \cup X.$$

Define $W_{\pm\pm} := W \cap \{\pm x > 0, \pm y > 0\}$ and $X_\pm := X \cap \{\pm x > 0\}$.
The refinement $\{W_{++}, W_{+-}, W_{-+}, W_{--}, X_+, X_-, Z_+, Z_-, Z_0\}$ is now a Whitney regular $\mathbb{Q}$-algebraic stratification of W.

Thank you for your attention!