Ambient Lipschitz geometry of definable surface germs

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A connected definable set $X \subset \mathbb{R}^n$ inherits two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y.
X is **normally embedded or LNE(modern terminology)** if these two metrics on X are equivalent.

A **surface germ** X is a closed two-dimensional germ at $\mathbf{0} \in \mathbb{R}^{\mathbf{n}}$ definable in a polynomially bounded o-minimal structure with the field of exponents $\mathbb{F}$.
Surface germs X and Y are **outer (inner) Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $X \to Y$.

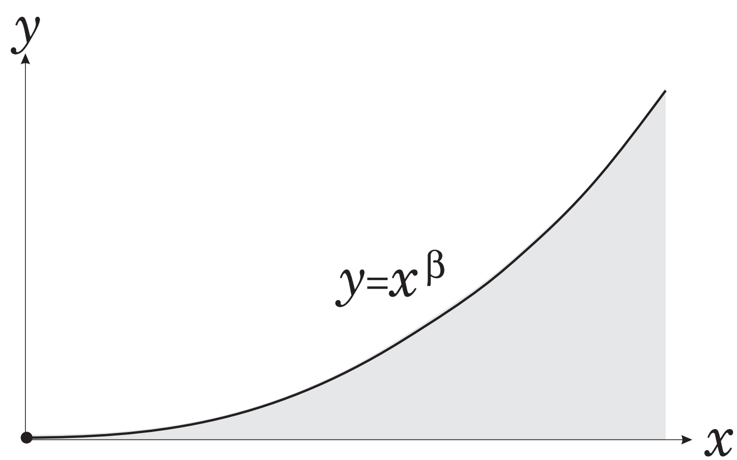
The germs are called ambient Lipschitz equivalent if there exists a germ of a bi-Lipschitz Homeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Psi(X) = Y$.

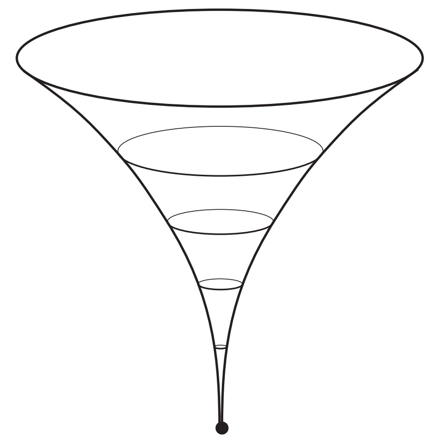
(The germs are called ambient Lipschitz equivalent if there exists a germ of a bi-Lipschitz Homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(X) = Y$.)

Finiteness theorems (Mostowski 85, Parusinski 94, Valette 05): Any definable family has finitely many outer Lipschitz equivalence classes.

For $\beta \in \mathbb{F}_{\geq 1}$, the **standard β-Hölder triangle** is the surface germ $T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}$.

The **standard β-horn** is $C_\beta = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$.





A **β-Hölder triangle** is a germ inner Lipschitz equivalent to T_β.

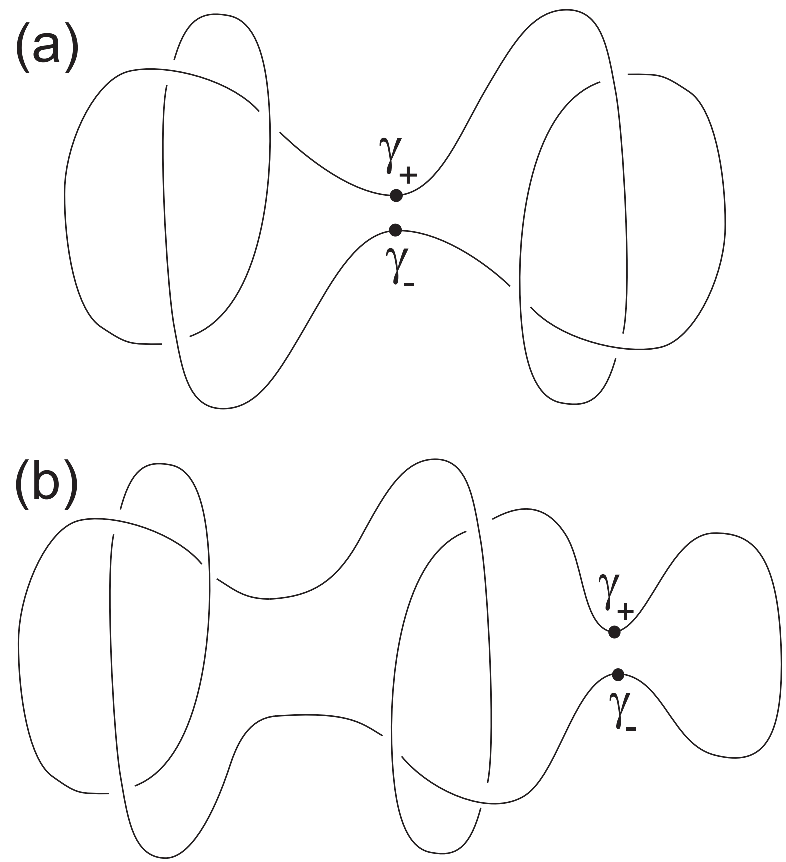
A **β-horn** is a germ inner Lipschitz equivalent to C_β.

First Question

Are Topologically equivalent and Outer Lipschitz Equivalent germs Ambient Equivalent?

Answer : **No!**

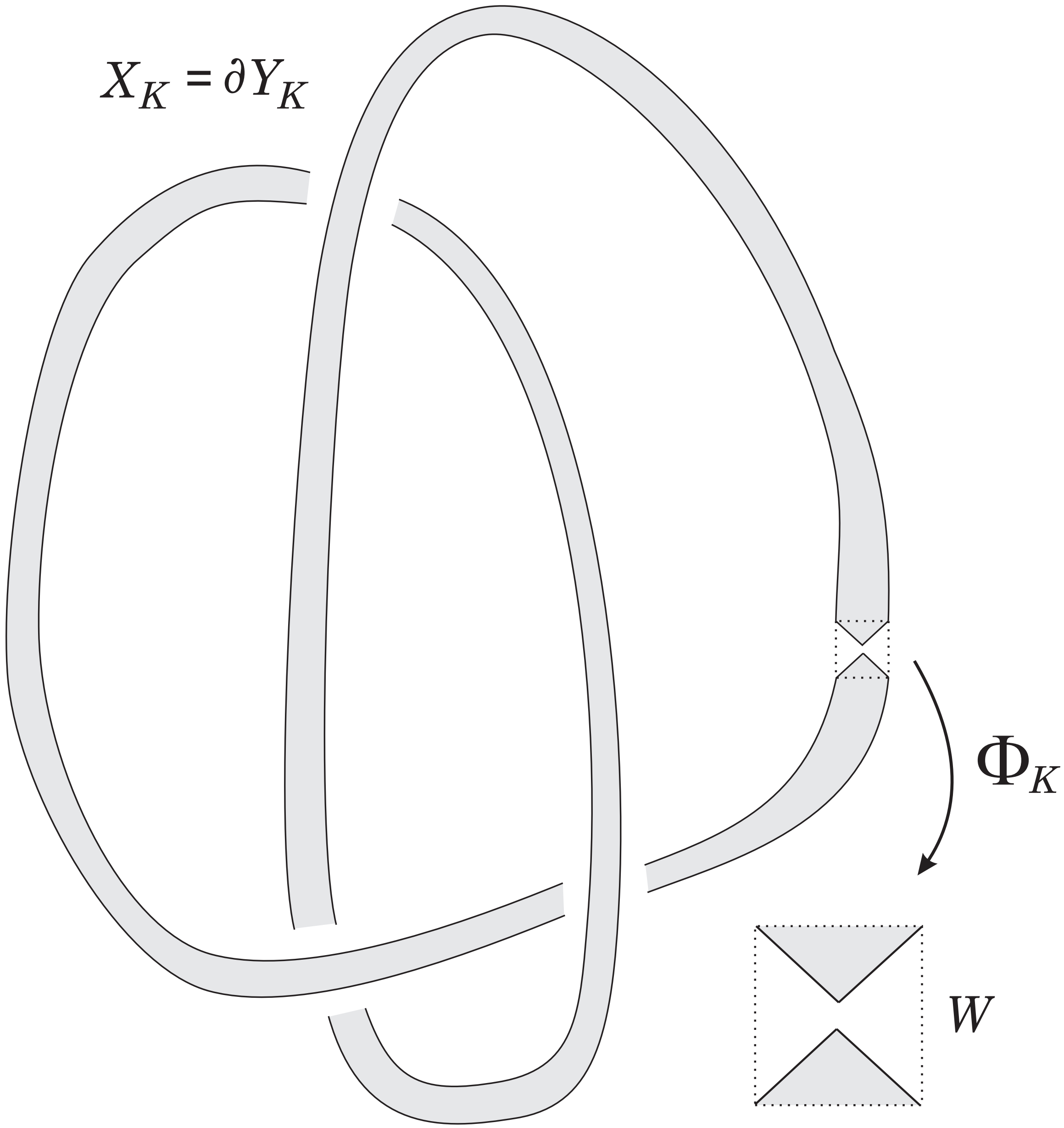
Examples (LB, A. Gabrielov):



Universality Theorem (LB, Brandenbursky, Gabrielov)

For any knot K, one can construct a germ of a surface X_K in $\mathbb{R}^4$ such that:
1. The link at the origin of X_K is a trivial knot;
2. The germs X_K are outer bi-Lipschitz equivalent for all K;
3. Two germs X_K and $X_{K'}$ are ambient bi-Lipschitz equivalent only if the knots K and K' are isotopic.

But, Notice in these examples X is **not!** LNE.


$X_K = \partial Y_K$
Φ_K
W

Natural question.

What about LNE surfaces?

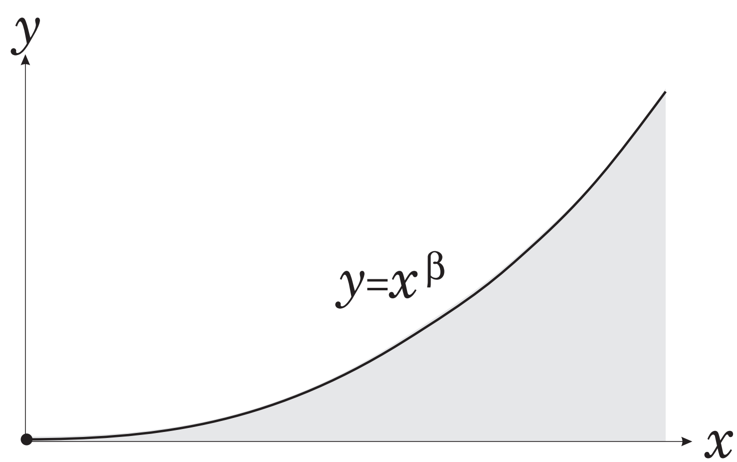
”Conjecture”: Two LNE surface germs are Ambient Lipschitz Equivalent if they are Topologically Equivalent and Outer Lipschitz Equivalent.

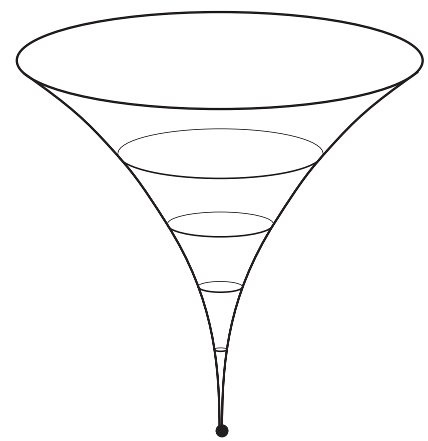
Actually the conjecture is true for surface germs in $\mathbb{R}^3$ with isolated singularities with connected links.

Theorem - combination (LB, D. Lopes Medeiros) and (LB, A. Fernandes, Z. Jelonek)

For any $\beta \in \mathbb{F}_{\geq 1}$, $n \in \mathbb{N}_{\geq 3}$ with $n \neq 4$, every LNE β-Hölder triangle $T \subset \mathbb{R}^n$ is ambient Lipschitz equivalent to the standard β-Hölder triangle in $\mathbb{R}^n$ and every LNE β-horn $H \subset \mathbb{R}^n$ is ambient Lipschitz equivalent to the standard β-horn $H \subset \mathbb{R}^n$.

Theorem 1 (LB, D. Lopes Medeiros) Let $X \subset \mathbb{R}^3$ be a germ of a LNE β-Hölder triangle or a germ of LNE β-horn $H \subset \mathbb{R}^3$. Then X is ambient Lipschitz equivalent to the standard β-Hölder triangle in $\mathbb{R}^n$ and every LNE β-horn $H \subset \mathbb{R}^3$ is ambient Lipschitz equivalent to the standard β-horn $H \subset \mathbb{R}^3$.





This result is not true for non-isolated singularities of for the surfaces with not connected link.

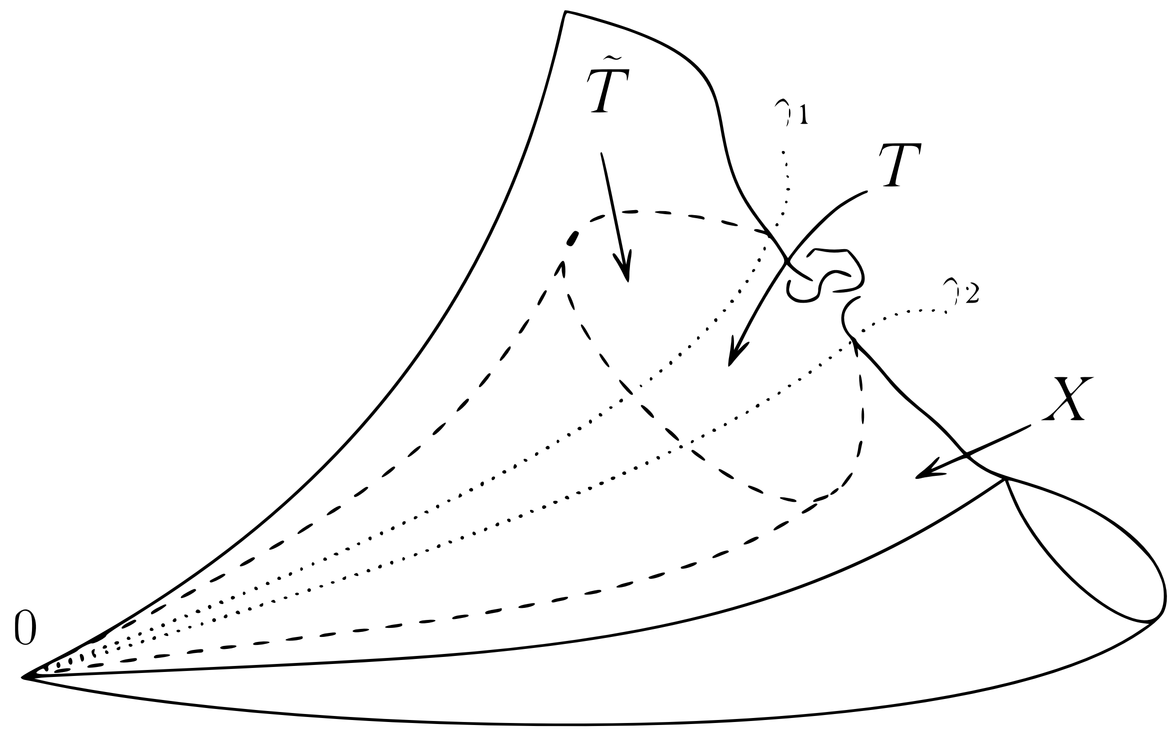
Theorem 2 (LB, A. Fernandes, Z. Jelonek)

Let $X, Y \subset \mathbb{R}^{k+1}$ be germs of a definable sets, $dim(X) = dim(Y) = k$. Suppose that X and Y are outer Lipschitz equivalent. Then they are ambient Lipschitz equivalent.

What about germs in R^4? LNE Hölder triangles.

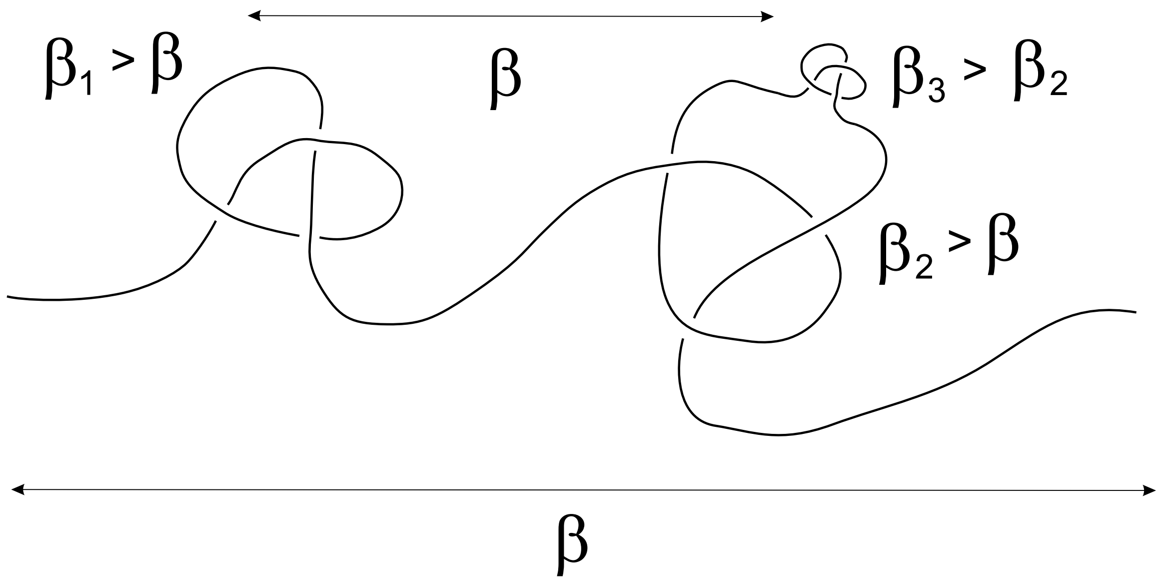
Universality Theorem 2 (LB, M.Denkowski, D Lopes Medeiros, E. Sampaio)

Fix $\beta \in \mathbb{F}_{>1}$. Then for any knot K, there exists an LNE Hölder triangle $T_{\beta,K} \subset \mathbb{R}^4$. Moreover, two triangles T_{β,K_1} and T_{β,K_2} of that kind are ambient bi-Lipschitz equivalent only if K_1 and K_2 are isotopic.



Next Question Ambient Lipschitz Classification of LNE Hölder triangles in $\mathbb{R}^4$.

What can happend?



Notice. All the vanishing exponents are rational.

The invariant graph Γ for a Hölder triangle in $\mathbb{R}^4$ is constructed as follows:

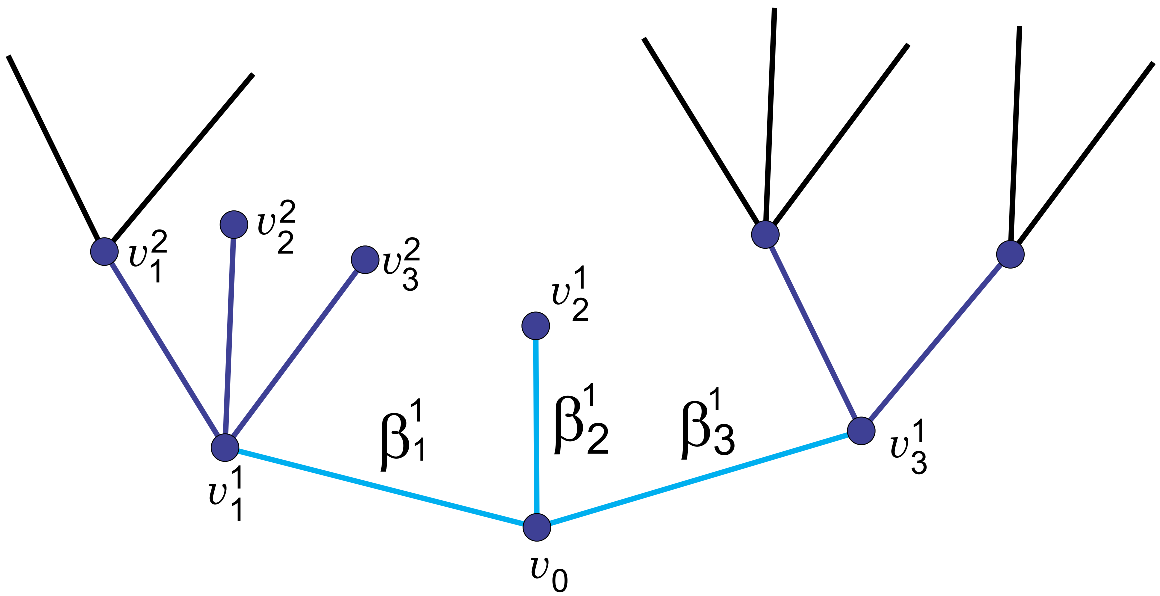
The vertex v_o correspond to the triangle.

The vertices ${v_1}^i$ correspond to the "primary" micro-knots ${K_1}^i$.

Each edge correspond to primary exponent.

The vertices ${v_2}^i$ correspond to the "secondary" micro-knots ${K_2}^i$.

The indices ${\beta_j}^i$ correspond to the micro-knot exponents.



v_1^2
v_2^2
v_3^2
v_2^1
β_1^1
β_2^1
β_3^1
v_1^1
v_3^1
v_0

Thank You.

Congratulations, Wiesław!