On arc-analytic geometry

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Definition

A function $f : X \to \mathbb{R}$ *is* arc-analytic *when* $f \circ \gamma$ *is analytic for every analytic arc* $\gamma : I \to X$*.*

Example 1.

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}\ , & x^2+y^2 \neq 0 \\ 0\ , & x^2+y^2 = 0 \end{cases}$$

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Example 2. The set

$$X = \{z(x^2+y^2) = x^3\} \setminus \{x^2+y^2 = 0, z \neq 0\}$$

is arc-symmetric (as the zero locus of an arc-analytic function).

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$$f(x,y,z,w) = \begin{cases} \frac{x^4}{x^2+y^2\sqrt{w^4+z^4}}\ , & \text{outside } y\text{−axis and } (z,w)\text{−plane} \\ 0\ , & \text{otherwise} \end{cases}$$

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f is arc-analytic as a quotient of two arc-analytic functions, that extends continuously to $\mathbb{R}^4$ (i.e., f is *arc-meromorphous* on $\mathbb{R}^4$).

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- Finer than Zariski topology (vide Cartan umbrella). In fact, $\mathscr{AR}$ topology is the finest among 'natural' Noetherian topologies containing Zariski topology (i.e., $\mathscr{AR}$, Nash, and regulous).

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Corollary (Nullstellensatz)

(i) *For every $\mathscr{AR}$-closed $X \subset \Omega$, $\mathcal{V}(\mathcal{I}(X)) = X$.*

(ii) *For every ideal I in $\mathscr{A}_a(\Omega)$, $\mathcal{I}(\mathcal{V}(I)) = \mathrm{Rad}(I)$.*

Definition

Let $n \geq 1$ and let $v_n : \mathbb{R}^n \to \mathbb{R}^n$ denote the semialgebraic map

$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \dots, \frac{x_n}{\sqrt{1 + x_n^2}} \right) .$$

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Since v_n is an analytic isomorphism onto the bounded open set $(-1, 1)^n$, it follows that globally subanalytic sets are subanalytic.

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Then, S is globally subanalytic in $\mathbb{R}^2$ as a bounded subanalytic set, but any arc-symmetric set in $\mathbb{R}^2$ containing S must contain the whole graph of the sine function as well. Thus, $\overline{S}^{\mathscr{AR}} = \mathbb{R}^2$.

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Following [Acquistapace et al., 2016], we say that a set $X \subset \Omega$ is *C-semianalytic*, when X is a union of a locally finite family of global basic semianalytic subsets of Ω, that is, sets of the form $\{f = 0, g_1 > 0, \dots, g_s > 0\}$, where $f, g_j \in \mathscr{A}(\Omega)$.

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Definition

Let $\mathscr{AR}_C(\Omega)$ denote the family of C-semianalytic sets $X \subset \Omega$ such that X is arc-symmetric in Ω and globally subanalytic as a subset of $\mathbb{R}^n$.

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We can recover Kurdyka's description of arc-symmetric sets in terms of connected components of desingularization:

Theorem

Let $X \in \mathscr{AR}_C(\Omega)$ be an $\mathscr{AR}_C$-irreducible set of dimension $k > 0$, and let $R \subset \Omega$ be its C-analytic closure. Let $\pi : \widetilde{R} \to R$ be a desingularization of R. Then, there exists a unique connected component $\widetilde{E}$ of $\widetilde{R}$ of dimension k, such that

$$\overline{\mathrm{Reg}_k(X)} \subset \pi(\widetilde{E}) \subset X\,.$$

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Let $X, Y \in \mathscr{AR}_C(\Omega)$, $Y \subsetneq X$, and suppose that X is $\mathscr{AR}_C$-irreducible of dimension $k > 0$. Then, $\dim Y < \dim X$.

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Corollary

The $\mathscr{AR}_C$-Krull dimension of an $\mathscr{AR}_C$-closed set is equal to its topological dimension.

Theorem 1

Let $X \in \mathscr{AR}_C(\Omega)$*. Then, there exists a globally subanalytic arc-symmetric function* $f : \Omega \to \mathbb{R}$*, such that* $X = f^{-1}(0)$*.*

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Theorem 2

Let $X \in \mathscr{AR}(\Omega)$ be a Nash subanalytic set. Then, there exists a continuous globally subanalytic function $f : \Omega \to \mathbb{R}$ and a simple normal crossings divisor $\Sigma \subset \Omega$, such that

(i) $\dim_x \Sigma \cap X < \dim_x X$, *for all* $x \in X$

(ii) f *is arc-analytic on* $\Omega \setminus \Sigma$, *and*

(iii) $X = f^{-1}(0)$.

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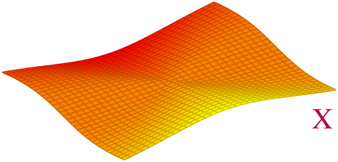

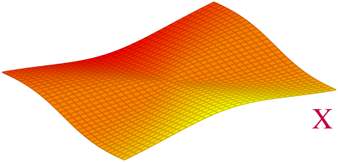
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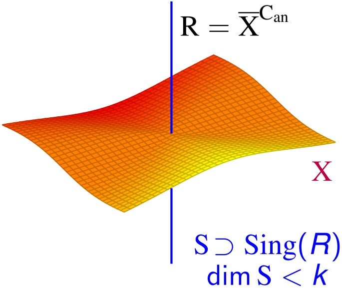
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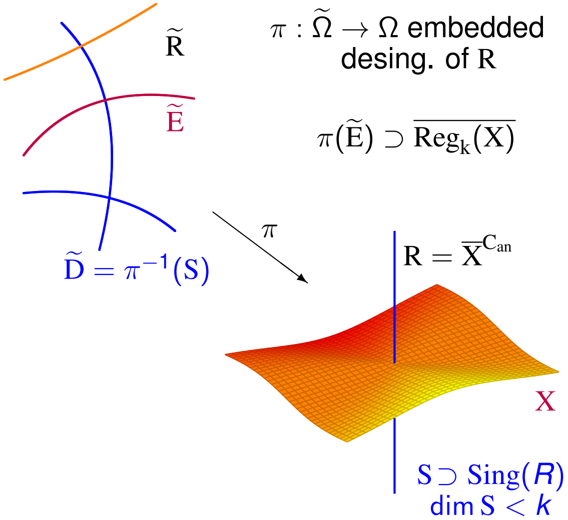
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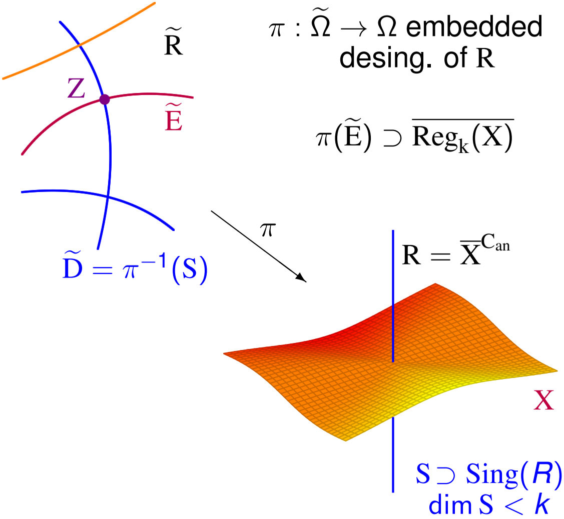

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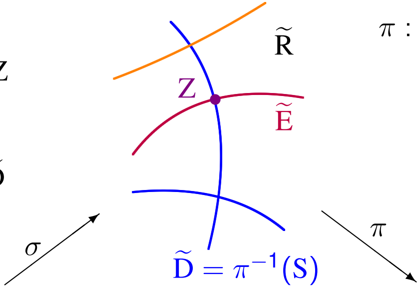

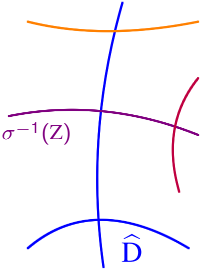

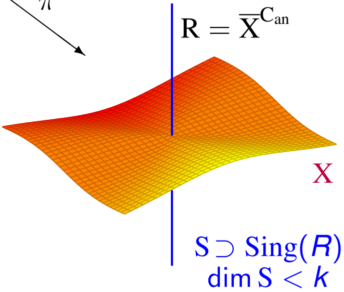

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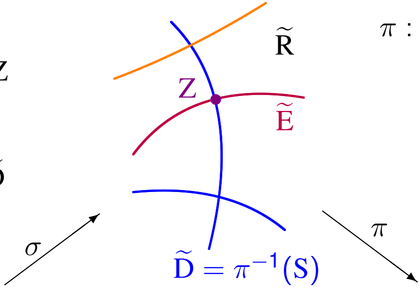

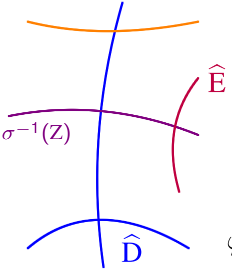

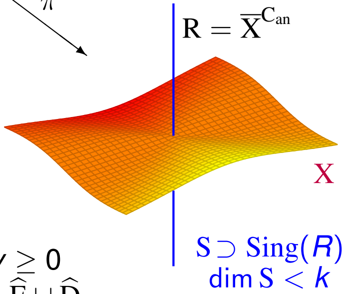

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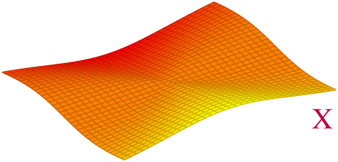
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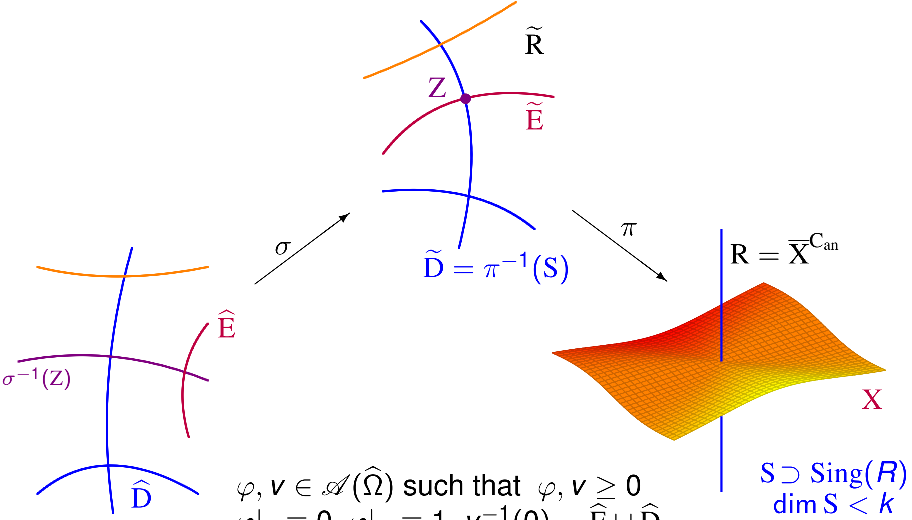

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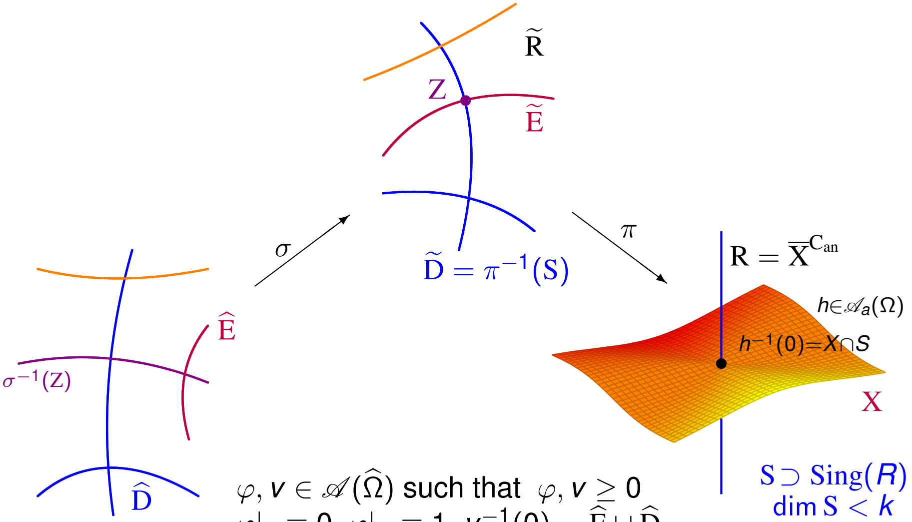

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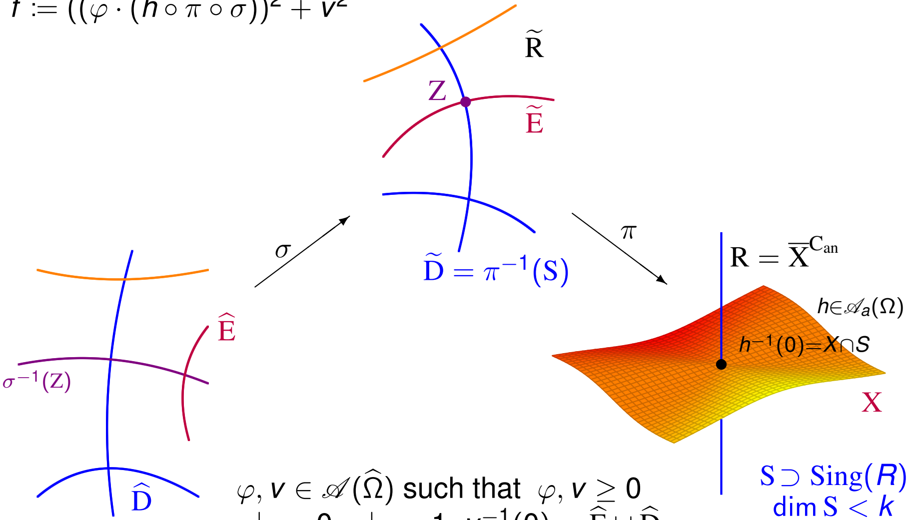
$$\widehat{f} := ((\varphi \cdot (h \circ \pi \circ \sigma))^2 + v^2$$



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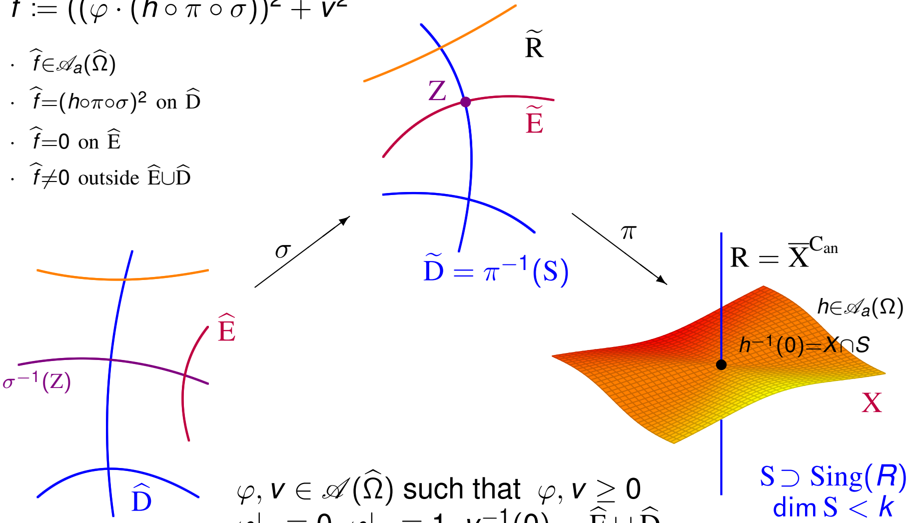
$\widehat{f} := ((\varphi \cdot (h \circ \pi \circ \sigma))^2 + v^2$

- $\widehat{f} \in \mathscr{A}_a(\widehat{\Omega})$
- $\widehat{f} = (h \circ \pi \circ \sigma)^2$ on $\widehat{\mathrm{D}}$
- $\widehat{f} = 0$ on $\widehat{\mathrm{E}}$
- $\widehat{f} \neq 0$ outside $\widehat{\mathrm{E}} \cup \widehat{\mathrm{D}}$



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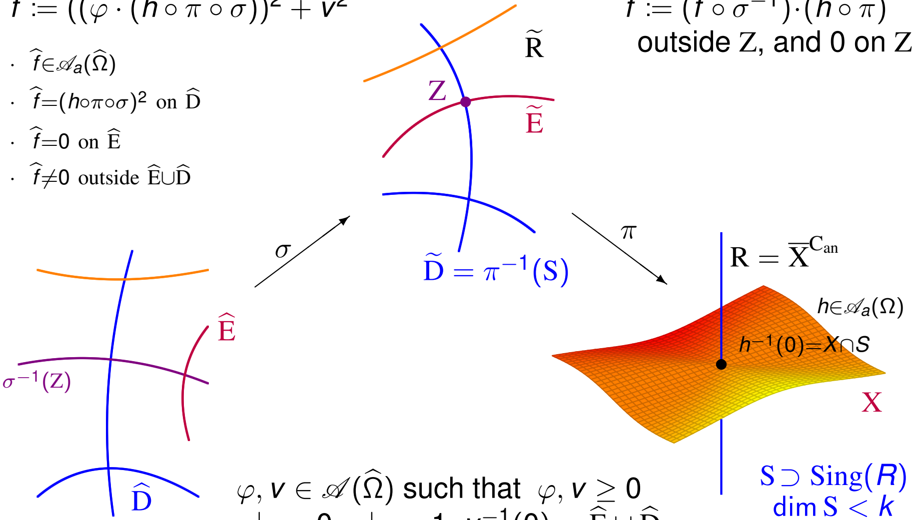
$\widetilde{f} := (\widehat{f} \circ \sigma^{-1}) \cdot (h \circ \pi)$ outside Z, and 0 on Z



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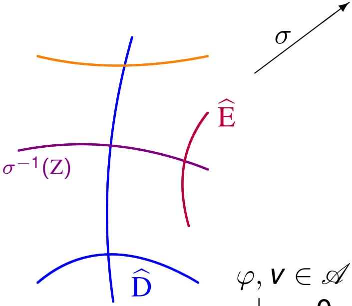
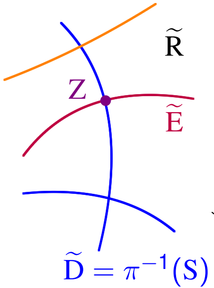
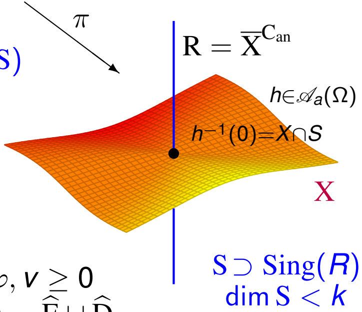
$f := \widetilde{f}\circ\pi^{-1}$ outside S

$f := h^3$ on S







$\varphi, v\in\mathscr{A}(\widehat{\Omega})$ such that $\varphi, v\geq 0$
$\varphi|_{\widehat{\mathrm{E}}}\equiv 0$, $\varphi|_{\widehat{\mathrm{D}}}\equiv 1$, $v^{-1}(0)=\widehat{\mathrm{E}}\cup\widehat{\mathrm{D}}$

$\widetilde{f}:\widetilde{\Omega}\to\mathbb{R}$ is arc-analytic:

Pick $\widetilde{\gamma}:I\to\widetilde{\Omega}$ analytic. Let $\widehat{\gamma}:I\to\widehat{\Omega}$ be its lifting by σ, so that $\sigma\circ\widehat{\gamma}=\widetilde{\gamma}$. We claim that

$$\widetilde{f}\circ\widetilde{\gamma}=(\widehat{f}\circ\widehat{\gamma})\cdot(h\circ\pi\circ\widetilde{\gamma})\,, \tag{1}$$

which implies that $\widetilde{f}\circ\widetilde{\gamma}$ is analytic.

Indeed, if $\widetilde{\gamma}(t)\notin Z$, then (1) holds because

$$(\widehat{f}\circ\sigma^{-1}\circ\widetilde{\gamma})(t)=(\widehat{f}\circ\sigma^{-1}\circ\sigma\circ\widehat{\gamma})(t)=(\widehat{f}\circ\widehat{\gamma})(t).$$

If, in turn, $\widetilde{\gamma}(t)\in Z$, then $(h\circ\pi\circ\widetilde{\gamma})(t)=0$, by definition of h, and hence both sides of (1) are equal to zero.

$f : \Omega \to \mathbb{R}$ is arc-analytic:

Pick $\gamma : I \to \Omega$ analytic. Let $\widetilde{\gamma} : I \to \widetilde{\Omega}$ and $\widehat{\gamma} : I \to \widehat{\Omega}$ be such that $\pi \circ \widetilde{\gamma} = \gamma$ and $\sigma \circ \widehat{\gamma} = \widetilde{\gamma}$. We claim that

$$f \circ \gamma = \widetilde{f} \circ \widetilde{\gamma}\,, \tag{2}$$

which implies that $f \circ \gamma$ is analytic. Indeed, if $\gamma(t) \notin S$, then (2) holds because $(\widetilde{f} \circ \pi^{-1} \circ \gamma)(t) = (\widetilde{f} \circ \pi^{-1} \circ \pi \circ \widetilde{\gamma})(t) = (\widetilde{f} \circ \widetilde{\gamma})(t)$. If, in turn, $\gamma(t) \in S \cap \pi(\widetilde{E})$, then $h(\gamma(t)) = 0$ and hence $(f \circ \gamma)(t) = 0$. But $\widetilde{\gamma}(t) \in Z$, and hence $(\widetilde{f} \circ \widetilde{\gamma})(t) = 0$ as well. Finally, if $\gamma(t) \in S \setminus \pi(\widetilde{E})$, then $\widetilde{\gamma}(t) \notin Z$ and $\widehat{\gamma}(t) \in \widehat{D}$; hence, by (1), we have

$$(\widetilde{f} \circ \widetilde{\gamma})(t) = ((\widehat{f} \circ \widehat{\gamma}) \cdot (h \circ \pi \circ \widetilde{\gamma}))(t) = \left(((h \circ \pi \circ \sigma)^2 \circ \widehat{\gamma}) \cdot (h \circ \pi \circ \widetilde{\gamma})\right)(t)$$

$$= \left((h \circ \pi \circ \widetilde{\gamma})^2 \cdot (h \circ \pi \circ \widetilde{\gamma})\right)(t) = (h \circ \gamma)^3(t) = (f \circ \gamma)(t)\,.$$

We have

$$\begin{aligned} f^{-1}(0) &= \{x \in \Omega \setminus S : (\widetilde{f} \circ \pi^{-1})(x) = 0\} \cup \{x \in S : h^3(x) = 0\} \\ &= \pi(\widetilde{E} \setminus \widetilde{D}) \cup (X \cap S) \\ &= \pi(\widetilde{E}) \cup (X \cap S) \\ &\supset \overline{\mathrm{Reg}_k(X)} \cup (X \cap S). \end{aligned}$$

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Since X was assumed $\mathscr{AR}_C$-irreducible, $f^{-1}(0)$ is $\mathscr{AR}_C$-closed and $\dim f^{-1}(0) = \dim X$, it follows that $f^{-1}(0) = X$. □

Recall:

Theorem 2

Let $X \in \mathscr{AR}(\Omega)$ be a Nash subanalytic set. Then, there exists a continuous globally subanalytic function $f : \Omega \to \mathbb{R}$ and a simple normal crossings divisor $\Sigma \subset \Omega$, such that

(i) $\dim_x \Sigma \cap X < \dim_x X$, *for all* $x \in X$

(ii) f *is arc-analytic on* $\Omega \setminus \Sigma$, *and*

(iii) $X = f^{-1}(0)$.

Conjecture

Every arc-symmetric globally subanalytic set is Nash subanalytic.

Thank you / Dziękuję!