Extension of k-regulous functions

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Preliminaries

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We say that a function $f : X \to \mathbb{R}$ *admits a rational representation*, if there exists a Zariski open and dense subset U of X, such that the restriction $f|_U$ is regular.

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Definition

We say that a function $f : X \to \mathbb{R}$ *admits a rational representation*, if there exists a Zariski open and dense subset U of X, such that the restriction $f|_U$ is regular. The ring of real valued functions on X which are of class $\mathcal{C}^k$ and admit rational representations is denoted by $\mathcal{R}_k(X)$.

Examples

Example

For $k \geq 0$, the function $f(x,y) = \begin{cases} \frac{x^{k+3}}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ belongs to $\mathcal{R}_k(\mathbb{R}^n)$, but it does not belong to $\mathcal{R}_{k+1}(\mathbb{R}^2)$.

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Example

Consider the Whitey umbrella $X = \{(x,y,z) \in \mathbb{R}^3 : x^2 - y^2 z = 0\}$. Let $f : X \to \mathbb{R}$ be given by

$$f(x,y,z) = \begin{cases} 0 & \text{if } z > 0, \\ e^{-\frac{1}{z^2}} & \text{if } z < 0. \end{cases}$$

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Then $f \in \mathcal{R}_k(X)$ for all $k \geq 0$, but f is not even semialgebraic.

k-regulous functions

Definition

Let $X \subset \mathbb{R}^n$ be an affine variety. A function $f : X \to \mathbb{R}$ is said to be *k-regulous*, if there exists a function $F \in \mathcal{R}_k(\mathbb{R}^n)$ such that $F|_X = f$. The ring of k-regulous functions on X is denoted by $\mathcal{R}^k(X)$.

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As seen in the example on the last slide, in general $\mathcal{R}_k(X) \not\subset \mathcal{R}^k(X)$. Fortunately, the other inclusion does hold:

Observation (Kollár, Nowak 2014)

For an affine variety X it holds that $\mathcal{R}^k(X) \subset \mathcal{R}_k(X)$.

Characterisation of 0-regulous functions

Theorem

Let $X \subset \mathbb{R}^n$ be an affine variety and let $f : X \to \mathbb{R}$ be a continuous function. Then, f is 0-regulous if and only if the following holds:

Characterisation of 0-regulous functions

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For every Zariski closed subset $Y \subset X$, the restriction $f|_Y$ admits a rational representation.

Moreover, if the condition is satisfied then there exists $F \in \mathcal{R}_0(\mathbb{R}^n)$ such that $F|_X = f$ and $F|_{\mathbb{R}^n \setminus X}$ is regular.

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Corollary 1

Let $X \subset \mathbb{R}^n$ be a nonsingular variety. Then $\mathcal{R}^0(X) = \mathcal{R}_0(X)$.

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Corollary 1

Let $X \subset \mathbb{R}^n$ be a nonsingular variety. Then $\mathcal{R}^0(X) = \mathcal{R}_0(X)$.

Corollary 2

Let $X \subset \mathbb{R}^n$ be a variety of dimension one. Then $\mathcal{R}^0(X) = \mathcal{R}_0(X)$.

What about $k > 0$?

Fix $k > 0$.

Question 1

Let $X \subset \mathbb{R}^n$ be a nonsingular variety. Does it follow that $\mathcal{R}^k(X) = \mathcal{R}_k(X)$?

Question 2

Let $X \subset \mathbb{R}^n$ be a variety of dimension one. Does it follow that $\mathcal{R}^k(X) = \mathcal{R}_k(X)$?

k-regulous functions on nonsingular varieties

Theorem (B., 2025)

Let $X \subset \mathbb{R}^n$ be a nonsingular affine variety and let $k \geq 0$. Then $\mathcal{R}^k(X) = \mathcal{R}_k(X)$.

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A rough sketch of the proof.

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Let $X \subset \mathbb{R}^n$ be a nonsingular affine variety and let $k \geq 0$. Then $\mathcal{R}^k(X) = \mathcal{R}_k(X)$.

A rough sketch of the proof.

Let $f \in \mathcal{R}_k(X)$. Assume first that there exists a regular retraction $r : \mathbb{R}^n \to X$, i.e. an n-tuple $r := (r_1, \ldots, r_n)$ of function in $\mathcal{R}(\mathbb{R}^n)$, such that $r(\mathbb{R}^n) = X$ and $r|_X = \mathrm{id}_X$.

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Proof of the theorem

A rough sketch of the proof.

In general, the best thing that we can say is that there exists a Euclidean neighbourhood U of X in $\mathbb{R}^n$ and a Nash retraction $r : U \to X$. Then $f \circ r$ is a function of class $\mathcal{C}^k$ on U.

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Let $Z \subset X$ be a proper algebraic subset of X, such that $f|_{X \setminus Z}$ is regular. Let $P, Q \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials, such that Q does not vanish at any point of $X \setminus Z$ and $f|_{X \setminus Z} = \frac{P|_{X \setminus Z}}{Q|_{X \setminus Z}}$.

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Now, using some algebra, we find a regular mapping $p : \mathbb{R}^n \to \mathbb{R}^n$, which interpolates r on $X \cup r^{-1}(Z)$ up to derivatives of some large order.

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Let $X \subset \mathbb{R}^n$ be an affine variety of dimension 1, which is a $\mathcal{C}^{k+1}$ submanifold of $\mathbb{R}^n$.

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$$f(x,y) := \begin{cases} \frac{y}{x} & \text{for } x \neq 0, \\ \varphi'(0) & \text{for } x = 0. \end{cases}$$

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By assumption, there exists $F \in \mathcal{R}^k(\mathbb{R}^2)$ with $F|_X = f$. Then $G := xF - y$ vanishes identically on X, and $\frac{\partial G}{\partial y}(0) = -1$. □

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Thanks to results of Fukui, Kurdyka and Paunescu, the implicit function theorem holds in the class of $\mathcal{C}^1$ arc-analytic functions. Since regulous functions are arc-analytic, it follows that k-regulous manifolds locally can be represented as graphs of arc-analytic functions. □

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Let $X \subset \mathbb{R}^n$ be a 1-regulous submanifold of dimension one. Then, it is actually an analytic submanifold of $\mathbb{R}^n$.

Corollary

Let $X \subset \mathbb{R}^n$ be an affine variety of dimension 1, which is also a $\mathcal{C}^{k+1}$-submanifold of $\mathbb{R}^n$. Assume that $\mathcal{R}^k(X) = \mathcal{R}_k(X)$. Then X is an analytic submanifold of $\mathbb{R}^n$.

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Example

Let $X := \{(x, y) \in \mathbb{R}^2 : x^3 = y^7\}$. Then $\mathcal{R}^1(X) \subsetneq \mathcal{R}_1(X)$.

About Question 2

It turns out that if X is a one dimensional analytic submanifold of $\mathbb{R}^n$, then the answer to Question 2 is affirmative:

Theorem

Let $X \subset \mathbb{R}^n$ be an affine variety of dimension one, which is a submanifold of class at least $\mathcal{C}^2$ in $\mathbb{R}^n$. Then, the following conditions are equivalent:

1. $\mathcal{R}^1(X) = \mathcal{R}_1(X)$,
2. $\mathcal{R}^k(X) = \mathcal{R}_k(X)$ for all k,
3. X is a 1-regulous submanifold of $\mathbb{R}^n$,
4. X is a k-regulous submanifold of $\mathbb{R}^n$ for all k,
5. X is an analytic submanifold of $\mathbb{R}^n$,
6. $\mathcal{R}^0(X) = \mathcal{R}^1(X) = \mathcal{R}^2(X) = \ldots$ (although $\mathcal{R}^0(X) \subsetneq \mathcal{R}(X)$ unless X is nonsingular).

Questions

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Let $X \subset \mathbb{R}^n$ be an affine curve, which is not an analytic submanifold of $\mathbb{R}^n$. What condition does one need to impose on $f \in \mathcal{R}_k(X)$ to make sure that it is k-regulous?

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Question

Let $X \subset \mathbb{R}^n$ be an affine variety, which is also a k-regulous submanifold of $\mathbb{R}^n$. Does there exist an intrinsic condition in the spirit of the one given by Kollár and Nowak, which characterises k-regulous functions on X?

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Remark

If X is of dimension greater than 1, then in general
X is a 1-regulous submanifold $\not\Leftrightarrow$ X is an analytic submanifold.

Thank you for your attention!

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