Differentiable approximation of continuous definable maps that preserves the image

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1. Differentiable approximation of continuous definable maps...

Weierstrass' approximation

Weierstrass' approximation

Every continuous map $f : X \to \mathbb{R}^m$, defined on a compact subset X of $\mathbb{R}^n$, can be uniformly approximated by polynomial maps.

Uniform norm

Let $X \subset \mathbb{R}^n$ be a **compact** subset and $Y \subset \mathbb{R}^m$ any subset. For a continuous map $f : X \to Y$ we consider the uniform norm:

$$\|f\| := \|f\|_\infty = \max_{x \in X} |f(x)|_m,$$

where $|\cdot|_m$ is the Euclidean norm of $\mathbb{R}^m$.

Weierstrass' approximation

$$f : X \to \mathbb{R}^m$$

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Several difficulties arise when one tries to restrict the image of the approximating map to a fixed target space $Y \subset \mathbb{R}^m$. For instance, there exist no non-constant polynomial maps from the 2-dimensional sphere $\mathbb{S}^2$ to the circle $\mathbb{S}^1$.

Strategy

(i) Consider, instead of polynomials maps, a more flexible class of approximating maps (like differentiable maps).

Whitney's approximation

Whitney's approximation

Let $Y \subset \mathbb{R}^m$ be a submanifold of class $\mathcal{C}^p$, with $p \geq 1$ or $p = \infty$. Then, each continuous map $f : X \to Y$ (defined on a compact subset X of $\mathbb{R}^n$), can be uniformly approximated by maps of class $\mathcal{C}^p$.

A map $f : X \to Y$ is **of class** $\mathcal{C}^p$ ($p \geq 1$ or $p = \infty$) if there exist an open neighborhood $U \subset \mathbb{R}^n$ of X and a map $F : U \to \mathbb{R}^m$ of class $\mathcal{C}^p$ in the usual sense, such that f is the restriction $F|_X$ of F to X.

Whitney's approximation - general target space

Example

Let $Y := \{\mathrm{xy} = 0\} \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^2$ an open neighborhood of Y. Suppose that there exists a $\mathcal{C}^1$ retraction $\pi : \Omega \to Y$. As π is the identity on Y, we have $d_0\pi = \mathrm{id}_{\mathbb{R}^2}$. Thus, π is a local diffeomorphism at the origin, which is a contradiction.

Strategy

(i) Consider, instead of polynomials maps, a more flexible class of approximating maps (like differentiable maps).

(ii) Consider domains of definitions and target spaces in some suitable tame category (like a fixed o-minimal structure).

We deal with: *Differentiable approximation of continuous definable maps...*

Convention

We consider a **fixed** o-minimal structure on the ordered field of real numbers $\mathbb{R}$. When we refer to definable sets or definable maps we mean definable in this fixed o-minimal structure.

A definable map $f : X \to Y$ is **of class** $\mathcal{C}^p$ if it is the restriction of a definable map $F : U \to \mathbb{R}^m$ of class $\mathcal{C}^p$ in the usual sense, where U is a definable open neighborhood of X in $\mathbb{R}^n$.

Triangulations of definable sets

Triangulations of definable sets

Let $X \subset \mathbb{R}^n$ be a compact definable set, then there exists a **definable triangulation** $(\mathcal{K}, \varphi)$ in $\mathbb{R}^n$ of X. That is, there exists a finite simplicial complex $\mathcal{K}$ of $\mathbb{R}^n$ and a definable homeomorphism $\varphi : |\mathcal{K}| \to X$.

Moreover, we can choose the definable triangulation $(\mathcal{K}, \varphi)$ **compatible** with a finite family $\mathcal{E}$ of definable subsets of X. That is, each $E \in \mathcal{E}$ is a union of some $\varphi(\mathring{\sigma})$ for some $\sigma \in \mathcal{K}$.

$\mathring{\sigma} :=$ Relative interior of σ

Differentiable approximation for semialgebraic maps

Theorem (Fernando, Ghiloni 2019)

Let $p \geq 1$ be an integer and $f : X \to Y$ be a continuous definable map between a compact definable sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. If the definable target space $Y \subset \mathbb{R}^m$ has a definable triangulation $(\mathcal{K}, \varphi)$ such that $\varphi : |\mathcal{K}| \to Y$ is of class $\mathcal{C}^p$, then f can be uniformly approximated by definable maps $g : X \to Y$ of class $\mathcal{C}^p$.

Remark: $p = 1, 2, \ldots$ but it is not allowed $p = \infty$.

The case $p = \infty$

Example

Let $X := [0,1]$ and $Y := \{\mathrm{xy} = 0\} \cap \{|x| \leq 1\} \cap \{|y| \leq 1\} \subset \mathbb{R}^2$. Consider the continuous semialgebraic map $f : X \to Y$ defined as follows:

$$f(x) := \begin{cases} \left(\frac{1}{2} - x, 0\right), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \left(0, x - \frac{1}{2}\right), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $g : X \to Y$ be any semialgebraic map of class $\mathcal{C}^\infty$. As g is analytic, then either $g(X) \subset \{\mathrm{x} = 0\}$ or $g(X) \subset \{\mathrm{y} = 0\}$. We deduce, $\|f - g\| \geq \frac{1}{2}$. Thus, f cannot be approximated by a $\mathcal{C}^\infty$ semialgebraic map.

Pawłucki's desingularization

Strict $\mathcal{C}^p$-refinement theorem (Pawłucki 2023)

Let $X \subset \mathbb{R}^n$ be a compact definable set and $f : X \to \mathbb{R}^m$ a continuous definable map. Let $(\mathcal{T}, \psi)$ be a definable triangulation compatible with a finite family $\mathcal{E}$ of definable subsets of X. Then, for each integer $p \geq 1$, there exists a definable triangulation $(\mathcal{K}, \varphi)$ of $|\mathcal{T}|$ such that:

- $\mathcal{K}$ is a refinement of $\mathcal{T}$.
- $\varphi(\sigma) = \sigma$ for each simplex $\sigma \in \mathcal{T}$.
- $(\mathcal{K}, \psi \circ \varphi)$ is a definable triangulation of X compatible with $\mathcal{E}$.
- $\psi \circ \varphi : |\mathcal{K}| \to X$ is of class $\mathcal{C}^p$.
- $\psi \circ \varphi|_{\mathring{\sigma}}$ is an embedding of class $\mathcal{C}^p$ for each $\sigma \in \mathcal{K}$.
- $f \circ \psi \circ \varphi : |\mathcal{K}| \to X$ is of class $\mathcal{C}^p$.

The case $p = \infty$

Example

Consider the continuous semialgebraic function $f : \mathbb{R} \to \mathbb{R}, t \mapsto |t|$. Assume that there exists a semialgebraic homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ such that $f \circ \varphi$ is of class $\mathcal{C}^\infty$. We may assume $\varphi(0) = 0$. As φ is analytic, its germ at 0 satisfies $\varphi \sim t^k$ with k a positive odd integer. Thus, $f \circ \varphi \sim |t|^k$, a contradiction.

Definable approximation

Definable approximation (Fernando-Ghiloni + Pawłucki 2023)

Let $p \geq 1$ be an integer. Each continuous definable map $f : X \to Y$ between a compact definable set $X \subset \mathbb{R}^n$ and a definable set $Y \subset \mathbb{R}^m$ can be uniformly approximated by definable maps $g : X \to Y$ of class $\mathcal{C}^p$.

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Let $f : X \to Y$ be a definable map. As f is definable, the set $f(X)$ is definable:

$$X \xrightarrow{f} f(X) \hookrightarrow Y$$

2. ...that preserves the image

Control on the image of the approximating map

Problem: Can we find an approximating map $g : X \to Y$ of class $\mathcal{C}^p$ such that $g(X) = f(X)$?

Definable $\mathcal{C}^p$-approximation that preserves the image

Theorem (2025)

Let $f : X \to \mathbb{R}^m$ be a continuous definable map defined on a compact definable set $X \subset \mathbb{R}^n$. Let $\varepsilon > 0$ and let $p \geq 1$ be an integer. Then, there exists a definable map $g : X \to \mathbb{R}^m$ of class $\mathcal{C}^p$ such that

- $\|f - g\| < \varepsilon$,
- $g(X) = f(X)$.

Main ingredients of the proof

- Pawłucki's desingularization
- Surjective simplicial approximation
- "(Definable) manipulations"

Simplicial approximation

Finite simplicial approximation

Let $\mathcal{K}$ be a finite simplicial complex of $\mathbb{R}^n$, $\mathcal{L}$ a finite simplicial complex of $\mathbb{R}^m$ and $f : |\mathcal{K}| \to |\mathcal{L}|$ a continuous [redacted] map. Then, for each $\varepsilon > 0$ there exist a subdivision $\mathcal{K}^*$ of $\mathcal{K}$, a subdivision $\mathcal{L}^*$ of $\mathcal{L}$ and a [redacted] simplicial map $h : |\mathcal{K}^*| \to |\mathcal{L}^*|$ such that $\|f - h\| < \varepsilon$.

Surjective simplicial approximation

Surjective finite simplicial approximation (2025)

Let $\mathcal{K}$ be a finite simplicial complex of $\mathbb{R}^n$, $\mathcal{L}$ a finite simplicial complex of $\mathbb{R}^m$ and $f : |\mathcal{K}| \to |\mathcal{L}|$ a continuous **surjective definable** map. Then, for each $\varepsilon > 0$ there exist a subdivision $\mathcal{K}^*$ of $\mathcal{K}$, a subdivision $\mathcal{L}^*$ of $\mathcal{L}$ and a **surjective** simplicial map $h : |\mathcal{K}^*| \to |\mathcal{L}^*|$ such that $\|f - h\| < \varepsilon$.

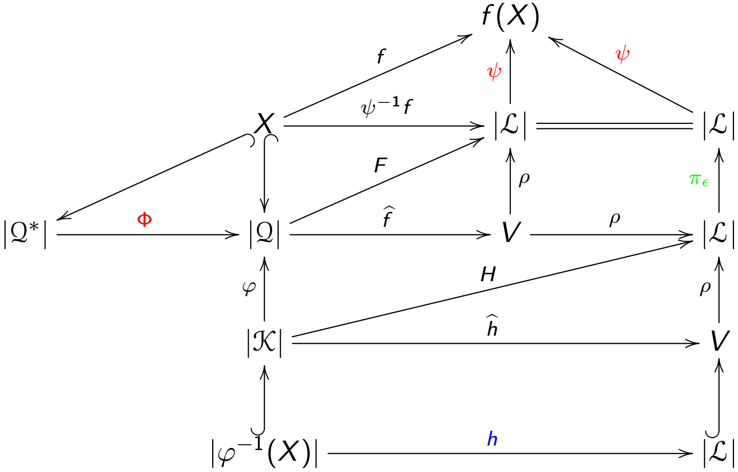
Surjective simplicial approximation

Example

Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes such that $|\mathcal{K}| = [0, 1]$ and $|\mathcal{L}| = [0, 1]^2$ and $f : |\mathcal{K}| \to |\mathcal{L}|$ any continuous surjective map.

Then no surjective simplicial approximation is possible because $|\mathcal{K}|$ has dimension 1 and $|\mathcal{L}|$ has dimension 2.

Sketch of the proof in a (non-commutative) diagram



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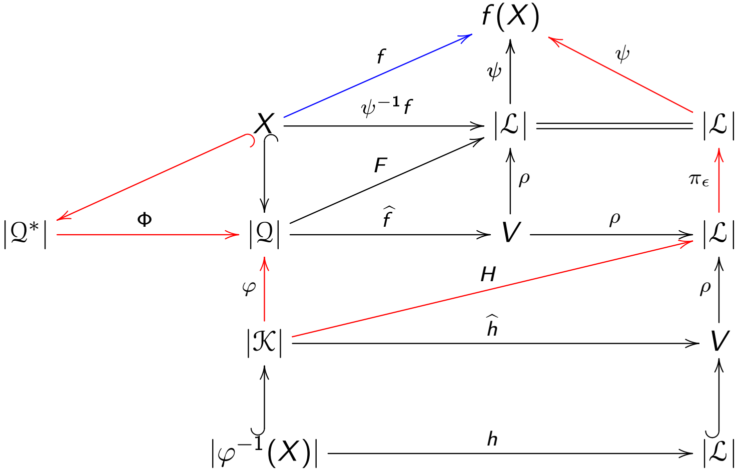

Thank you!