Pawłucki's contributions to subanalytic geometry

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Wiesław Pawłucki's habilitation thesis [1986]

The non-semianalytic points of a subanalytic set $X \subset \mathbb{R}^n$ form a closed subanalytic subset of X.

The motivation and techniques are related to two basic properties of semianalytic sets:

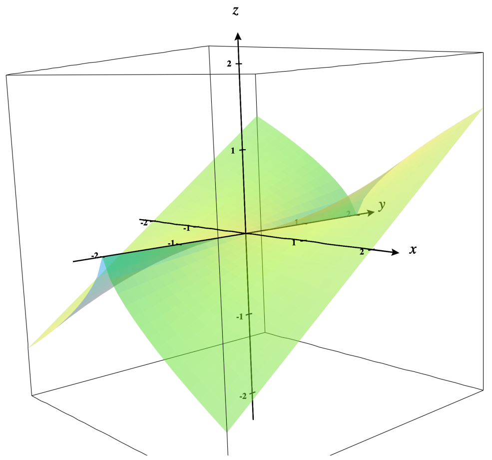
(1) Every semianalytic set lies locally in a real analytic set of the same dimension.

(2) A stratified version of the coherence property of complex analytic sets.

Examples

(1) Osgood [1916] The image of (x, xy, xye^y) lies in no proper analytic subset of $\mathbb{R}^3$.

(2) Real algebraic sets already needn't be coherent.



$X: z^3 - x^2y^3 = 0$

At a nonzero point b of the x-axis, the ideal $\mathcal{A}_b(X)$ of germs of analytic functions vanishing on X is not generated by $z^3 - x^2y^3$, but by the Nash function $z - x^{2/3}y$.

Understanding of these phenomena in the work of Pawłucki, BM, etc., is related to understanding of the behaviour of local algebraic invariants of subanalytic sets.

(1) Gabrielov [1971]

An analytic mapping $y = \varphi(x)$ induces homomorphisms of analytic and formal local rings, for any point a of the source:

$$\varphi_a^* : \mathcal{O}_b \to \mathcal{O}_a \cong \mathbb{R}\{x - a\}$$
$$\hat{\varphi}_a^* : \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a \cong \mathbb{R}[\![x - a]\!]$$

where $b = \varphi(a)$. Set

$$r_a(\varphi) := \text{generic rank of } \varphi \text{ at } a$$
$$r_a^{\mathcal{F}}(\varphi) := \dim \widehat{\mathcal{O}}_b / \mathrm{Ker}\, \hat{\varphi}_a^*$$
$$r_a^{\mathcal{A}}(\varphi) := \dim \mathcal{O}_b / \mathrm{Ker}\, \varphi_a^*$$

Then $r_a(\varphi) \le r_a^{\mathcal{F}}(\varphi) \le r_a^{\mathcal{A}}(\varphi)$.

In Osgood's example, $r_a = 2$, $r_a^{\mathcal{F}} = r_a^{\mathcal{A}} = 3$.

Theorem [Gabrielov 1973]. The following are equivalent.

(a) $r_a(\varphi) = r_a^{\mathcal{F}}(\varphi)$

(b) $r_a(\varphi) = r_a^{\mathcal{A}}(\varphi)$ (φ is regular at a)

(c) Composite function property, $\mathcal{O}_a \cap \hat{\varphi}_a^* \widehat{\mathcal{O}}_b = \varphi_a^* \mathcal{O}_b$.

Pawłucki's thesis depends on a parametrized version of Gabrielov's theorem. Gabrielov's ranks correspond to 3 notions of local dimension of a closed subanalytic subset $X \subset \mathbb{R}^n$:

$$\begin{aligned} d_b(X) &:= \dim_b(X) \\ d_b^{\mathcal{F}}(X) &:= \dim \widehat{\mathcal{O}}_b / \mathcal{F}_b(X) \\ d_b^{\mathcal{A}}(X) &:= \dim \mathcal{O}_b / \mathcal{A}_b(X) \end{aligned}$$

where $\mathcal{A}_b(X)$ and $\mathcal{F}_b(X)$ are the analytic and formal local ideals of X at a point b.

$$\mathcal{A}_b(X) = \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \varphi_a^*, \qquad \mathcal{F}_b(X) := \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \hat{\varphi}_a^*$$

where $\varphi : M \to \mathbb{R}^n$ is a proper analytic mapping with image X.

Theorem [Pawłucki 1992]. $\{a \in M : \varphi \text{ is not regular at } a\}$ is a proper closed analytic subset of M.

φ is regular if and only if $X = \varphi(M)$ is a Nash subanalytic set (i.e., locally a finite union of pure dimensional subanalytic sets each lying in an analytic set of the same dimension).

Corollary. The set of non-Nash points of a subanalytic set X form a subanalytic subset of codimenion ≥ 2.

The result on non-semianalytic points follows.

(2) Semicoherence

Is every subanalytic set $X \subset \mathbb{R}^n$ semicoherent [BM 1987]?

X is (formally) semicoherent if it has a stratfication $X = \cup X_i$ such that every point of $\overline{X_i}$ admits a neighbourhood V with finitely many parametrized formal power series

$$f_{ij}(b, y) = \sum_{\alpha \in \mathbb{N}^n} f_{ij,\alpha}(b)(y-b)^\alpha \in \mathbb{R}[\![y-b]\!]$$

generating $\mathcal{F}_b(X)$, $b \in X_i \cap V$, where the coefficients $f_{ij,\alpha}$ are analytic functions on $X_i \cap V$ which are subanalytic.

Nash subanalytic sets are semicoherent [BM 1987].

[Hironaka 1986]: Every subanalytic set X is semicoherent (formally and analytically); therefore, X has a stratification such that $d_b^{\mathcal{F}}(X)$, $d_b^{\mathcal{A}}(X)$ are constant on strata.

Counterexample of Pawłucki [1989]

Given $\{a_n\} \subset I = (-\delta, \delta) \subset \mathbb{R}$, there is an analytic mapping

$$\Phi(u, w, t) = (u,\, t,\, tw,\, t\varphi(u, w),\, t\psi(u, w, t)),$$

$(u, w, t) \in I^3$, where Φ has no formal relation (i.e., $\mathrm{Ker}\,\widehat{\Phi}_a^* = 0$) precisely at the points $a = (a_n, 0, 0)$, and Φ has a convergent relation throughout any open interval in $I \backslash \{a_n\}$. For example:

(a) If $\lim a_n = 0$ but no $a_n = 0$: the image X (of a compact neighbourhood of 0) is neither $\mathcal{F}$- nor $\mathcal{A}$-semicoherent.

(b) If $\{a_n\}$ is dense in I: X is $\mathcal{A}$- but not $\mathcal{F}$-semicoherent. (Does $\mathcal{F}$-semicoherent $\implies$ $\mathcal{A}$-semicoherent?)

(c) If the accumulation points of $\{a_n\}$ form a convergent sequence: the points where X is not semicoherent do not form a subanalytic subset.

The class of semicoherent subanalytic sets is characterized by several remarkably equivalent tameness properties.

Theorem [BMP 1996]. X is semicoherent if and only if

$$\mathcal{C}^\infty(X) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X)\,.$$

The development of this idea is related to problems on composition and extension of differentiable functions of origin in Whitney [1930s–40s], Glaeser [1950s–60s]. For example:

Theorem [Whitney 1943]. Every $\mathcal{C}^{2k}$ even function $f(x)$ $(k \le \infty)$ can be written $f(x) = g(x^2)$, where g is $\mathcal{C}^k$.

The loss of differentiability is related to Chevalley's lemma. A formal power series $G(y)$ vanishes to order k if $F(x) = G(x^2)$ vanishes to order $2k$. (Chevalley estimate $\ell(k) := 2k$.)

Theorem [BM 1987–97, BMP 1996]. The following conditions are equivalent.

(1) X is semicoherent.

(2) Chevalley estimate, uniform with respect to $b \in X$.

(3) The Hilbert-Samuel function $b \mapsto H_{X,b} \in \mathbb{N}^{\mathbb{N}}$, where

$$H_{X,b}(k) := \dim_{\mathbb{R}} \frac{\widehat{\mathcal{O}}_b}{\mathcal{F}_b(X) + \widehat{\mathfrak{m}}_b^{k+1}}\,,$$

is upper-semicontinuous in the subanalytic Zariski topology.

(4) X has the $\mathcal{C}^\infty$ composite function property.

(5) $\mathcal{C}^\infty(X) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X)$.

Composite function problem

Given a proper real analytic mapping $\varphi : M \to \mathbb{R}^n$, how to recognize whether $f \in \mathcal{C}^\infty(M)$ can be expressed as $f = g \circ \varphi$, where $g \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Necessary formal condition. For any $b \in X := \varphi(M)$, there is $G_b \in \widehat{\mathcal{O}}_b$ such that the Taylor expansion $\hat{f}_a = \hat{\varphi}_a^*(G_b)$, for all $a \in \varphi^{-1}(b)$.

Say φ has the composite function property if this is sufficient. The composite function property depends only on $X = \varphi(M)$.

The analogous $\mathcal{C}^k$ composite function property ($k < \infty$) holds for any closed subanalytic X, with a certain loss of differentiability [BMP 1996].

To find a solution $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ of the composite problem involves extension of the pointwise formal solutions G_b mod $\mathcal{F}_b(X)$, $b \in X$, to a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$.

Whitney's extension problem

How to recognize whether a function $f : X \to \mathbb{R}$ defined on a closed subset X of $\mathbb{R}^n$ is the restriction of a $\mathcal{C}^k$ function?

Whitney [1934] in the case $n = 1$;

Fefferman [2006]: necessary and sufficient criterion, building of work of Glaeser [1958] and BMP [2003].

Geometric extension problem. Suppose f is semialgebraic (or definable), and f extends to a $\mathcal{C}^k$ function on $\mathbb{R}^n$. Does f extend to a semialgebraic (or definable) $\mathcal{C}^k$ function?

Aschenbrenner, Thamrongthanyalak [2019] in the case $k = 1$;

Fefferman, Luli [2022] in the case $n = 2$;

B, Campesato, M [2021] in the general case, with a certain loss of differentiability (related to [BMP 1996]).

The definable extension problem is important also in the context of the classical Whitney extension theorem.

A $\mathcal{C}^k$ Whitney field on a closed subset $X \subset \mathbb{R}^n$ is a parametrized family of polynomials

$$\sum_{|\alpha| \le k} \frac{f_\alpha(a)}{\alpha!}(x-a)^\alpha, \quad a \in X,$$

where the coefficients $f_\alpha \in \mathcal{C}^0(X)$ satisfy

$$f_\alpha(y) - \sum_{|\beta| \le k-|\alpha|} \frac{f_{\alpha+\beta}(x)}{\beta!}(y-x)^\beta = o(|x-y|^{k-|\alpha|})$$

as $|x-y| \to 0$, $x, y \in X$.

Theorem [Kurdyka, Pawłucki 1997, 2014]. Given a subanalytic (or definable) $\mathcal{C}^k$ Whitney field on a closed subset $X \subset \mathbb{R}^n$, and $m \ge k$, there is a subanalytic function $f \in \mathcal{C}^k(\mathbb{R}^n)$, such that $D^\alpha f = f_\alpha$ on X, $|\alpha| \le k$, and $f \in \mathcal{C}^m(\mathbb{R}^n \backslash X)$.

In the semialgebraic case, there is an extension which is Nash on $\mathbb{R}^n \backslash X$ [Kocel-Cynk, Pawłucki, A. Valette 2019].

The proof uses Λ_p-regular cell decomposition, which involves estimates of Yomdin [1987] and Gromov [1987] from their work on uniform $\mathcal{C}^r$ parametrization of semialgebraic or definable sets. The latter is developed in work of Pila-Wilkie [2006], Binyamini-Novikov [2019], as well as by Kocel-Cynk, Pawłucki, Valette [2018].

Uniform $\mathcal{C}^r$ parametrization

Theorem. Consider a semialgebraic (or definable) family of closed subsets $X \subset [0,1]^n$ of dimension k

(e.g., semialgebraic sets defined by finitely many polynomials p_j with $\sum \deg p_j \le d$).

Let $r \in \mathbb{N}$. Then every X can be covered by $\mathcal{C}^r$ semialgebraic mappings $\varphi_1, \ldots, \varphi_m : [0,1]^k \to \mathbb{R}^n$, such that the number of mappings m and the partial derivatives $D^\alpha \varphi_i$, $|\alpha| \le r$, are bounded by constants depending only on (n, k, r) (and d).

This can be regarded as a uniform $\mathcal{C}^r$ version of Hironaka's rectilinearization theorem. There is no $\mathcal{C}^\infty$ analogue, even in the semialgebraic case.

Strict $\mathcal{C}^r$ triangulation

Let us conclude with Pawłucki's remarkable recent work on strict $\mathcal{C}^r$ triangulation, where the parametrization is a homeomorphism.

Theorem [P 2024]. Given a closed semialgebraic (or definable) subset $X \subset \mathbb{R}^n$ and $r \in \mathbb{N}$, there is a finite simplicial complex $\Sigma \subset \mathbb{R}^n$, and a definable $\mathcal{C}^r$ mapping $h : U \to \mathbb{R}^n$ from a neighbourhood U of Σ, such that h restricts to a homeomorphism $\Sigma \to X$, and h induces a $\mathcal{C}^r$ embedding of every open simplex in Σ.

Moreover, given a definable continuous mapping $f : X \to \mathbb{R}^p$ and a finite family of definable subsets $X_i \subset X$, the parametrization h can be constructed so that $f \circ h$ is $\mathcal{C}^r$, and each $h^{-1}(X_i)$ is a union of open simplices.

Thank you for listening,

and warmest wishes to Wiesław for many happy, healthy and productive years ahead!