EQUATIONS FOR UNKNOWN C^m FUNCTIONS

(JOINT WORK WITH G. K. LULI)

THIS TALK IS ABOUT

LINEAR EQUATIONS

FOR

UNKNOWN C^m FUNCTIONS

$$A_1 F_1 + \cdots + A_D F_D = f$$

A_i & f GIVEN FNS. ON $\mathbb{R}^n$.
(MAYBE POLYNOMIALS)

$F_1, \dots, F_D$ ARE THE UNKNOWNS.

WANT $F_1, \dots, F_D \in C^m(\mathbb{R}^n)$

WANT $F_1, \dots, F_D \in C^m(\mathbb{R}^n)$

m IS FIXED, GIVEN

WE'RE NOT ALLOWED TO LOSE DERIVATIVES

THIS SUBJECT STARTED WITH THE C^0 CASE,

IN WORKS OF

$$\left.\begin{array}{l}\text{BRENNER}\\ \text{EPSTEIN}\\ \text{HOCHSTER}\\ \text{KOLLÁR}\end{array}\right\}\ \text{ALGEBRAIC GEOMETERS}$$

MOTIVATION

KOLLÁR ASKED:

HOW CAN WE TELL WHEN A RATIONAL FN. P/Q IS CONTINUOUS?

(Maybe P & Q have Common Zeros)

$F = P/Q$ CONTINUOUS

MEANS THAT THE LINEAR EQ.

$QF = P$

HAS A C^0 SOLUTION F.

BRENNER, EPSTEIN, HOCHSTER
WERE INTERESTED IN
NOTIONS OF CLOSURE OF IDEALS

GIVEN POLYS $A_1, \dots, A_D$ on $\mathbb{R}^n$,

THE POLY'S f FOR WHICH

$$A_1 F_1 + \cdots + A_D F_D = f$$

ADMITS A <u>CONTINUOUS</u> SOLUTION

$F_1, \dots, F_D \in C^0$

FORM AN <u>IDEAL</u> IN THE RING OF POLY'S

AN EXAMPLE

DUE TO BRENNER - EPSTEIN - HOCHSTER:

$$\boxed{x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f}$$

on $\mathbb{R}^3$.

THE QUESTIONS

<u>Q1</u>: FIX m, n, D.

GIVEN FNS. $A_1, \dots, A_D, f$ on $\mathbb{R}^n$

(MAYBE POLY'S)

$$\boxed{\begin{array}{l}\text{DECIDE WHETHER THERE EXIST} \\ F_1, \dots, F_D \in C^m(\mathbb{R}^n) \text{ s.t.} \\ A_1 F_1 + \dots + A_D F_D = f.\end{array}}$$

Q2: Fix m, n, D.

Let $A_1, \cdots, A_D$ be <u>POLY'S</u> on $\mathbb{R}^n$.

The poly's f s.t. the EQUATION $A_1F_1 + \cdots + A_DF_D = f$ admits a C^m solution form an ideal in the ring of poly's on $\mathbb{R}^n$.

EXHIBIT GENERATORS!

Q3 : Fix m, n, D.

Suppose $A_1, \cdots, A_D, f$ are Poly's on $\mathbb{R}^n$.

Suppose the eq. $A_1 F_1 + \cdots + A_D F_D = f$ admits a C^m solution $F_1, \cdots, F_D$.

Can we take the C^m fns F_i to be

RATIONAL FNS?

SEMIALGEBRAIC FNS?

WE CAN ANSWER
QUESTIONS Q1 & Q2.

WE HAVE ONLY PARTIAL RESULTS
ON
Q3.

HOW I CAME TO GET INVOLVED

EMAIL FROM JANOS KOLLÁR -

Do I know anything about these problems?

My immediate answer:

NO!

THEN I THOUGHT FURTHER
and REALIZED,
TO MY GREAT SURPRISE
that the answer is

YES!

THERE'S A JOINT PAPER BY KOLLÁR & ME, GIVING 2 INDEPENDENT SOLUTIONS (!) OF Q1 IN THE C^0-CASE.

THE B-E-H-K PROBLEMS
ARE CLOSELY RELATED TO
A PROBLEM I'VE SPENT
YEARS THINKING ABOUT,
NAMELY ...

WHITNEY'S EXTENSION PROBLEM

WHITNEY'S PROBLEM:

Fix m, n.

Let $\varphi : E \to \mathbb{R}$

($E \subset \mathbb{R}^n$ COMPACT, OTHERWISE ARBITRARY)

DECIDE WHETHER $\exists F \in C^m(\mathbb{R}^n)$

S.T.

$F = \varphi$ on E.

THE TWO QUESTIONS

$$\boxed{\exists?\, F \in C^m(\mathbb{R}^n) \text{ s.t. } F = \varphi \text{ on } E}$$

and

$$\boxed{\begin{array}{l}\exists?\, F_1, \ldots, F_D \in C^m(\mathbb{R}^n) \text{ s.t.} \\ A_1 F_1 + \cdots + A_D F_D = f\end{array}}$$

are both special cases of a basic question about **BUNDLES**

WILL DEFINE

BUNDLES

FIBERS

SUBBUNDLES

SECTIONS OF BUNDLES

Then will pose the

MAIN QUESTION ON BUNDLES

and explain its solution in terms

of

GLAESER REFINEMENT

That will solve Q1
from BRENNER-EPSTEIN-HOCHSTER-KOLLÁR.

If time permits, will discuss
Solution of Q2

Will say at most a few words re Q3

PREPARING TO DEFINE BUNDLES

NOTATION: Fix m, n, D.

$\mathcal{P}$ = VECTOR SPACE of polys of degree $\leq m$ on $\mathbb{R}^n$

$J_x F$ = m^{th} DEGREE TAYLOR POLY OF F AT x

So $J_x F \in \mathcal{P}$.

JETS OF VECTOR-VALUED FNS.

$C^m(\mathbb{R}^n, \mathbb{R}^D)$ DENOTES THE SPACE OF $\mathbb{R}^D$-VALUED C^m FUNCTIONS

$$\vec{F} = (F_1, \dots, F_D) \text{ on } \mathbb{R}^n.$$

For such $\vec{F}$, we define

$$J_x\vec{F} = (J_x F_1, \dots, J_x F_D).$$

JET MULTIPLICATION

$$P, Q \in \mathcal{P} \Rightarrow P \odot_x Q \equiv J_x(PQ).$$

So $J_x(FG) = J_x(F) \odot_x J_x(G)$.

$\mathcal{R}_x = (\mathcal{P}, \odot_x)$ is a **RING**

(the "ring of m-jets at x")

$$R_x^D = \underbrace{R_x \oplus R_x \oplus \cdots \oplus R_x}_{D \text{ COPIES}}$$

ELEMENTS $(P_1, \ldots, P_D) \in R_x^D$

MAY BE MULTIPLIED BY $Q \in R_x$:

$$Q \odot_x (P_1, \ldots, P_D) \equiv (Q \odot_x P_1, \ldots, Q \odot_x P_D).$$

So $\boxed{R_x^D \text{ is an } R_x\text{-MODULE}}$

WILL BE LOOKING AT

$\mathcal{R}_x$ - SUBMODULES OF $\mathcal{R}_x^D$

(WILL SEE EXAMPLES IN A FEW MINUTES)

THE FOLLOWING DEGENERATE CASES ARE ALLOWED:

- All of $\mathcal{R}_x^D$
- $\{0\}$, the ZERO SUBMODULE
- ϕ, THE EMPTY SET (!)

NOW WE CAN DEFINE

$$\boxed{\begin{array}{l}\text{BUNDLES}\\ \text{FIBERS}\\ \text{SUBBUNDLES}\\ \text{SECTIONS}\end{array}}$$

Let $E \subset \mathbb{R}^n$ be compact.

A **Bundle** is a family

$\mathcal{H} = (H_x)_{x \in E}$,

where, for each x,

$\boxed{H_x \text{ is a translate (coset) of an } \mathcal{R}_x\text{-submodule of } \mathcal{R}_x^D.}$

So $H_x = f_x + I_x$ ← SUBMODULE

ELEMENT of $\mathcal{R}_x^D$

NOTE, ANY PARTICULAR H_x MAY BE EMPTY, BECAUSE WE ALLOWED $\emptyset$ AS AN R_x-submodule of R_x^D.

If $\mathcal{H} = (H_x)_{x \in E}$ is a bundle,

then E is the **BASE** of $\mathcal{H}$,

and H_x is the **FIBER** of $\mathcal{H}$ at x

Let $\mathcal{H} = (H_x)_{x \in E}$, $\tilde{\mathcal{H}} = (\tilde{H}_x)_{x \in E}$ be two bundles with the same base E.

Then $\tilde{\mathcal{H}}$ is a **SUBBUNDLE** of $\mathcal{H}$

$\iff$ $H_x \supseteq \tilde{H}_x$ for all $x \in E$.

We write $\mathcal{H} \supset \tilde{\mathcal{H}}$ to denote this condition.

LET $\mathcal{H} = (H_x)_{x \in E}$ be a BUNDLE.

Let $\vec{F} \in C^m(\mathbb{R}^n, \mathbb{R}^D)$.

WE SAY THAT $\boxed{\vec{F} \text{ is a SECTION of } \mathcal{H}}$

$$\Updownarrow$$

$$\boxed{J_x \vec{F} \in H_x \text{ for all } x \in E.}$$

MAIN QUESTION ABOUT BUNDLES

GIVEN A BUNDLE $\mathcal{H}$,
DECIDE WHETHER $\mathcal{H}$
HAS A SECTION.

TRIVIAL REMARK:

If $\mathcal{H}$ has even one EMPTY FIBER, then it has NO SECTIONS.

THE WHITNEY EXTENSION PROB.

and

Q1 RE BRENNER-EPSTEIN-HOCHSTER-Kollár

are special cases of the

MAIN QUESTION ABOUT BUNDLES.

WHITNEY PROB.

GIVEN $\varphi : E \to \mathbb{R}$.

$\exists ?$ $F \in C^m(\mathbb{R}^n)$ s.t. $F = \varphi$ on E

For $x \in E$, set $H_x = \{ P \in \mathcal{P} : P(x) = \varphi(x) \}$.

Then $\mathcal{H} = (H_x)_{x \in E}$ is a bundle,
and a SECTION of that bundle is precisely
a fn $F \in C^m(\mathbb{R}^n)$ s.t. $F = \varphi$ on E.

So $\exists F \in C^m(\mathbb{R}^n)$ s.t. $F = \varphi$ on E

$$\Updownarrow$$

$\mathcal{H}$ has a section

Brenner-Epstein-Hochster-Kollár Q1:

$\exists ?\ F_1, \ldots, F_D \in C^m(\mathbb{R}^n)$ s.t. $A_1F_1 + \cdots + A_DF_D = f$

For $x \in \mathbb{R}^n$, set

$$H_x = \{(P_1, \ldots, P_D) \in \mathcal{P}^D : A_1(x)P_1(x) + \cdots + A_D(x)P_D(x) = f(x)\}.$$

Then $\mathcal{H} = (H_x)_{x \in \mathbb{R}^n}$ is a bundle, and a section of $\mathcal{H}$ is precisely a fn. $\vec{F} = (F_1, \ldots, F_D) \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ s.t. $A_1F_1 + \cdots + A_DF_D = f$.

So $\exists\, F_1, \dots, F_D \in C^m(\mathbb{R}^n)$ s.t.

$A_1 F_1 + \dots + A_D F_D = f$

$\Updownarrow$

$\mathcal{H}$ has a section.

(WE'VE CHEATED — $\mathbb{R}^n$ ISN'T COMPACT,

so, technically, $\mathcal{H}$ isn't a bundle.

That technicality is easily overcome.

DETAILS OMITTED)

So FAR, WE'VE

DEFINED BUNDLES

and posed the

MAIN PROBLEM re BUNDLES.

We've seen that the Whitney ext. prob

& Brenner-Epstein-Hochster-Kollár Q1

are special cases.

Now we sketch the
SOLUTION
to the
MAIN PROB re BUNDLES.

JOINT WORK WITH

GARVING (KEVIN) LULI,

building on decades of previous work by

H. Whitney,
G. Glaeser
Y. Brudnyi & P. Shvartsman
E. Bierstone, P. Milman and W. Pawłucki

MAIN QUESTION:

How can we tell whether a given bundle has a section?

To ANSWER the MAIN QUESTION, we INTRODUCE the GLAESER REFINEMENT

GIVEN A BUNDLE

$$\mathcal{H} = (H_x)_{x \in E},$$

WE WILL DEFINE ITS GLAESER REFINEMENT

$$\tilde{\mathcal{H}} = (\tilde{H}_x)_{x \in E}.$$

KEY PROPERTIES OF $\tilde{\mathcal{H}}$

- $\tilde{\mathcal{H}}$ IS A SUBBUNDLE of $\mathcal{H}$
- $\mathcal{H}$ AND $\tilde{\mathcal{H}}$ HAVE THE SAME SECTIONS
- $\tilde{\mathcal{H}}$ CAN BE COMPUTED FROM $\mathcal{H}$ by doing linear algebra in fixed dimension, and then taking a limit.

So, if we want to know whether

$\mathcal{H}$ has a section,

we may as well ask whether

$\tilde{\mathcal{H}}$ has a section.

(We pass from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ by discarding JUNK)

Even if all the fibers of $\mathcal{H}$

are NON-EMPTY,

it may happen that $\tilde{\mathcal{H}}$ has

some empty fibers.

In that case, $\tilde{\mathcal{H}}$ has no sections,

so

$\mathcal{H}$ has no sections.

ITERATED GLAESER REFINEMENT

Given a bundle $\mathcal{H}$, define

$$\mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots$$

by setting

$$\mathcal{H}_0 = \mathcal{H} \quad \text{and}$$

$\mathcal{H}_{\ell+1}$ = GLAESER REFINEMENT of $\mathcal{H}_\ell$.

WE CAN COMPUTE EACH $\mathcal{H}_\ell$.

For each ℓ, the sections of $\mathcal{H}_\ell$ are the same as the sections of $\mathcal{H}$.

THE PROCESS STABILIZES:

$$\mathcal{H}_{\ell_*} = \mathcal{H}_{\ell_*+1} = \mathcal{H}_{\ell_*+2} = \cdots$$

for $\ell_* = \ell_*(m, n, D)$

ANSWER TO THE MAIN QUESTION ON BUNDLES:

Let $\mathcal{H}$, $\mathcal{H}_{\ell_*}$ be as above.

Then $\mathcal{H}$ has a section

$\iff$ $\mathcal{H}_{\ell_*}$ has NO EMPTY FIBERS

So we've ANSWERED the MAIN QUESTION re BUNDLES, except that we haven't defined Glaeser refinement.

Will do so in the next few slides.

MOTIVATION

Let $F \in C^m(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$.

Given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$\forall x, y \in B(x_0, \delta)$, we have

$$|\partial^\alpha (J_x F - J_y F)(y)| \leq \varepsilon |x-y|^{m-|\alpha|}, \text{ all } |\alpha| \leq m.$$

(RHS $\equiv 0$ if $|x-y| = m - |\alpha| = 0$)

(It's just Taylor's thm.)

WE FIX A LARGE ENOUGH INTEGER CONST.

$$k = k(m, n, D).$$

DEFINITION OF GLAESER REFINEMENT

Let $\mathcal{H} = (H_x)_{x \in E}$ be a bundle.

Then the GLAESER REFINEMENT

$$\tilde{\mathcal{H}} = (\tilde{H}_x)_{x \in E}$$

IS DEFINED BY SETTING EACH FIBER

$\tilde{H}_{x_0}$ AS FOLLOWS:

$\tilde{H}_{x_0}$ consists of all $P_0 \in H_{x_0}$ s.t.

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$\forall x_1, \dots, x_k \in B(x_0, \delta) \cap E$

there exist $P_1 \in H_{x_1}, \dots, P_k \in H_{x_k}$

satisfying

$$|\partial^\alpha (P_i - P_j)(x_j)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \leq m$ and $i, j = 0, 1, \dots, k$.

NOTE: $P_0, P_1, \dots, P_k$ all appear in the definition, but P_0 stays fixed, while $P_1, \dots, P_k$ vary.

So we've defined the Glaeser refinement.
Let's check the KEY PROPERTIES
asserted above.

KEY PROPERTY 1:

$\tilde{\mathcal{H}}$ IS A SUBBUNDLE OF $\mathcal{H}$.

OF COURSE:

$\mathcal{H} = (H_x)_{x \in E}$, $\tilde{\mathcal{H}} = (\tilde{H}_x)_{x \in E}$

$\tilde{H}_{x_0}$ consists of all $P_0 \in H_{x_0}$ s.t. ——— .

So, obviously,

$$H_{x_0} \supset \tilde{H}_{x_0} .$$

KEY PROPERTY 2:

$\mathcal{H}$ & $\tilde{\mathcal{H}}$ HAVE THE SAME SECTIONS.

Since $\mathcal{H} \supset \tilde{\mathcal{H}}$, EVERY SECTION OF $\tilde{\mathcal{H}}$ is a section of $\mathcal{H}$.

MUST SHOW: Every section of $\mathcal{H}$ is a section of $\tilde{\mathcal{H}}$.

OK, suppose F is a section of $\mathcal{H}$.

So $F \in C^m$, and $J_x F \in H_x$ for all $x \in E$.

Must show $J_{x_0} F \in \tilde{H}_{x_0}$ for all $x_0 \in E$.

That is, MUST SHOW:

$$\boxed{\begin{array}{l} \text{Given } \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x_1, \dots, x_k \in B(x_0, \delta) \cap E, \\ \exists P_1 \in H_{x_1}, \dots, P_k \in H_{x_k} \text{ s.t.} \\ |\partial^\alpha (P_i - P_j)(x_j)| \le \varepsilon |x_i - x_j|^{m-|\alpha|} \quad (|\alpha| \le m,\ 0 \le i,j \le k) \end{array}}$$

where $P_0 \equiv J_{x_0} F$.

To show that, we just set

$$P_j = J_{x_j} F \quad \text{for } j = 1, \cdots, k.$$

Then $P_1 \in H_{x_1}, \cdots, P_k \in H_{x_k}$ because F is a section of $\mathcal{H}$.

The inequality

$$|\partial^\alpha (P_i - P_j)(x_j)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}$$

is just Taylor's thm, as noted above.

So we've proven

<u>KEY PROPERTY 2</u> :

$\mathcal{H}$ and $\tilde{\mathcal{H}}$ have the same sections.

KEY PROPERTY 3: $\tilde{\mathcal{H}}$ CAN BE COMPUTED FROM $\mathcal{H}$.

$\mathcal{H} = (H_x)_{x \in E}$.

Given $x_0, x_1, \dots, x_k \in E$ and $P_0, P_1, \dots, P_k \in \mathcal{P}^D$, set

$$Q(P_0, P_1, \dots, P_k, x_0, x_1, \dots, x_k) =$$

$$\sum_{x_i \neq x_j} \sum_{|\alpha| \leq m} \left(\frac{\partial^\alpha (P_i - P_j)(x_j)}{|x_i - x_j|^{m-|\alpha|}} \right)^2.$$

So $Q(P_0, P_1, \cdots, P_k, x_0, x_1, \cdots, x_k)$

is a quadratic form in $P_0, \cdots, P_k$

for fixed $x_0, x_1, \cdots, x_k$.

DEFINE

$$\mathcal{Q}(P_0, x_0, x_1, \cdots, x_k) =$$

$$\min\{Q(P_0, P_1, \cdots, P_k, x_0, x_1, \cdots, x_k) : P_1 \in H_{x_1}, P_2 \in H_{x_2}, \cdots, P_k \in H_{x_k}\}.$$

$\mathcal{Q}$ is the MIN of a quadratic form over an affine space, so we can compute it.

A GIVEN $P_0 \in H_{x_0}$ BELONGS TO $\tilde{H}_{x_0}$

if & only if

$$\boxed{\begin{array}{l} \mathcal{Q}(P_0, x_0, x_1, \dots, x_k) \to 0 \\ \quad \text{as } (x_1, \dots, x_k) \in E \times E \times \dots \times E \\ \qquad \text{tends to } (x_0, x_0, \dots, x_0). \end{array}}$$

We can decide this by doing linear alg. in fixed dimension (to compute $\mathcal{Q}$) and taking a limit as above.

So we've
DEFINED THE GLAESER REFINEMENT,
CHECKED ITS 3 KEY PROPERTIES,
and USED IT TO STATE THE ANSWER
TO THE QUESTION:

$\boxed{\text{How to decide whether a given bundle has a section.}}$

We haven't discussed the proof that our answer is correct.

That proof is long & hard.

Let's not say anything more about it.

ON TO BRENNER-EP.-HOCH.-KOLLÁR Q2:

Let $A_1, \cdots, A_D$ be polys.

The polys f expressible in the form

$$A_1 F_1 + \cdots + A_D F_D = f$$

with $F_1, \cdots, F_D \in C^m(\mathbb{R}^n)$

form an IDEAL in the ring of POLY'S on $\mathbb{R}^n$.

<u>PROBLEM:</u> EXHIBIT GENERATORS