AGAIN, THE RESULTS
ARE JOINT WORK
WITH
GARVING (KEVIN) LULI

Observe: Can't solve this problem by analysis alone, because it's about an ideal in a polynomial ring

Can't solve this problem by algebra alone, because it's about C^m functions

PLAN:

Make a clean splitting into a problem (99%) ENTIRELY ABOUT ANALYSIS,

and a problem (99%) ENTIRELY ABOUT ALGEBRA.

THE ANALYSIS PROBLEM:

GIVEN POLY'S $A_1, \dots, A_D$ on $\mathbb{R}^n$, CHARACTERIZE THE C^∞ FUNCTIONS f for which the equation

$$A_1 F_1 + \cdots + A_D F_D = f$$

has a C^m solution $\vec{F} = (F_1, \dots, F_D)$.

EXAMPLE :

$$x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f \quad (f \in C^\infty).$$

A <u>CONTINUOUS</u> solution F_1, F_2, F_3 exists

$\iff$

$$\boxed{\begin{array}{l} f = \partial_x f = \partial_y f = 0 \quad \text{for } x = y = 0 \\ \& \\ \partial_x \partial_y \partial_z f = 0 \quad \text{at } x = y = z = 0. \end{array}}$$

NOTE, 3rd DERIVATIVES of f ENTER, EVEN THOUGH WE ONLY ASK FOR CONTINUOUS F_1, F_2, F_3.

Solution of the Analysis Prob. in the General Case:

Given poly's $A_1, \cdots, A_D$ on $\mathbb{R}^n$, there exist linear differential operators $L_1, \cdots, L_K$ with the following properties:

(PROPERTIES A, B, C)

Each L_ν has the form

$$\boxed{L_\nu f(x) = \sum_{|\alpha| \le r} a_{\nu\alpha}(x)\, \partial^\alpha f(x)}$$

where r is a large integer

and the

$a_{\nu\alpha}(x)$ are semialgebraic functions.

↑

PROPERTY A

The equation

$$A_1 F_1 + \cdots + A_D F_D = f \qquad (f \in C^\infty)$$

admits a C^m solution $\vec{F} = (F_1, \ldots, F_D)$

$\Longleftrightarrow$ $L_\nu f = 0$ for each ν

PROPERTY B

$\boxed{\text{We can compute the coefficients } a_{\nu\alpha}(x) \text{ of the operators } L_\nu.}$

PROPERTY C

EXAMPLE:

For the EQUATION $x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f$

(for unknown CONTINUOUS F_1, F_2, F_3),

the differential operators L_ν are:

$$\boxed{\mathbb{1}_{x=y=0}\,\partial^0, \quad \mathbb{1}_{x=y=0}\,\partial_x, \quad \mathbb{1}_{x=y=0}\,\partial_y, \quad \text{and} \quad \mathbb{1}_{x=y=z=0}\,\partial_x \partial_y \partial_z}$$

The proof of the above result is based on further study of GLAESER REFINEMENTS.

THE ALGEBRA PROBLEM

GIVEN LINEAR DIFFERENTIAL OP'S L_ν, of the form

$$\sum_{|\alpha| \le r} a_{\nu\alpha}(x)\, \partial^\alpha, \quad (\nu = 1, \dots, \nu_{MAX})$$

SEMIALGEBRAIC

DEFINE AN IDEAL IN THE RING OF POLY'S AS FOLLOWS:

$I(L_1, \dots, L_{\nu_{MAX}})$ consists of all poly's P

s.t.

$L_\nu(QP) = 0$, all ν and all poly's Q.

PROBLEM: Given $L_1, \dots, L_{\nu_{MAX}}$,

EXHIBIT GENERATORS FOR

$I(L_1, \dots, L_{MAX})$.

Observe:

If the poly's P satisfying

$L_\nu P = 0 \quad (\nu = 1, \dots, \nu_{max})$

already form an ideal, then that ideal is precisely

$I(L_1, \dots, L_{\nu_{max}})$.

So, if we solve the above problem of computational algebra, then we can read off generators for the ideal of polys f s.t.

$$A_1 F_1 + \cdots + A_D F_D = f$$

admits a C^m solution $F_1, \ldots, F_D$.

IN FACT:

Let f be a poly. Then $f \in C^\infty$, so

$$A_1 F_1 + \dots + A_D F_D = f$$

has a C^m solution

$$\Longleftrightarrow$$

$$L_\nu f = 0 \quad (\nu = 1, \dots, \nu_{max})$$

$$\Longleftrightarrow L_\nu (Qf) = 0, \text{ all } \nu, \text{ all } Q$$

$$\Longleftrightarrow f \in \underbrace{I(L_1, \dots, L_{\nu max})}_{\text{WE CAN FIND GENERATORS}}.$$

SO MUCH FOR

BR-EP-HO-KO PROBLEM Q2.

ON TO Q3 !

RECALL Q3:

LET $A_1, \cdots, A_D, f$ BE POLYS.

SUPPOSE $\exists F_1, \cdots, F_D \in C^m(\mathbb{R}^n)$ S.T.

$$A_1 F_1 + \cdots + A_D F_D = f.$$

$$\boxed{\text{Can we take those } C^m \text{ fns } F_i \text{ to be rational? Semialgebraic?}}$$

A COUNTEREXAMPLE DUE TO

KOLLÁR & NOVAK

SHOWS THAT ONE

<u>CANNOT</u>

IN GENERAL TAKE THE F_i

to be RATIONAL FNS

CONJECTURE:

WE CAN ALWAYS TAKE THE F_i TO BE SEMIALGEBRAIC FNS.

[KOLLÁR & NOVAK PROVE IT FOR THE C^0 CASE.]

IS IT TRUE FOR C^m?

A MORE GENERAL CONJECTURE

CONCERNS

SEMIALGEBRAIC BUNDLES,

I.E.

$$\boxed{\begin{array}{l} \text{BUNDLES } \mathcal{H} = (H_x)_{x \in E} \quad \text{S.T.} \\ \{(x, P) \in E \times \mathcal{P}^D : P \in H_x\} \\ \text{IS A SEMIALGEBRAIC SET.} \end{array}}$$

CONJECTURE:

LET $\mathcal{H}$ BE A SEMIALGEBRAIC BUNDLE.

IF $\mathcal{H}$ HAS A SECTION,

THEN IT HAS A SEMIALGEBRAIC SECTION.

THAT CONJECTURE WAS PROVEN

BY G. K. LULI & ME

FOR BUNDLES $\mathcal{H} = (H_x)_{x \in E}$

WITH $E \subset \mathbb{R}^2$.

THE GENERAL CASE $(E \subset \mathbb{R}^n)$

IS OPEN & LOOKS HARD

THANKS
FOR
LISTENING!