A converse to Cartan's Theorem B: The extension property for real analytic and Nash sets

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Tame geometry and extensions of functions - Pawłucki 70

1. Introduction

1.1. Analytic case

Let $\Omega \subset \mathbb{R}^n$ be an open set. A subset $X \subset \Omega$ has the *analytic extension property* if each analytic function $f : X \to \mathbb{R}$ extends to an analytic function on Ω.

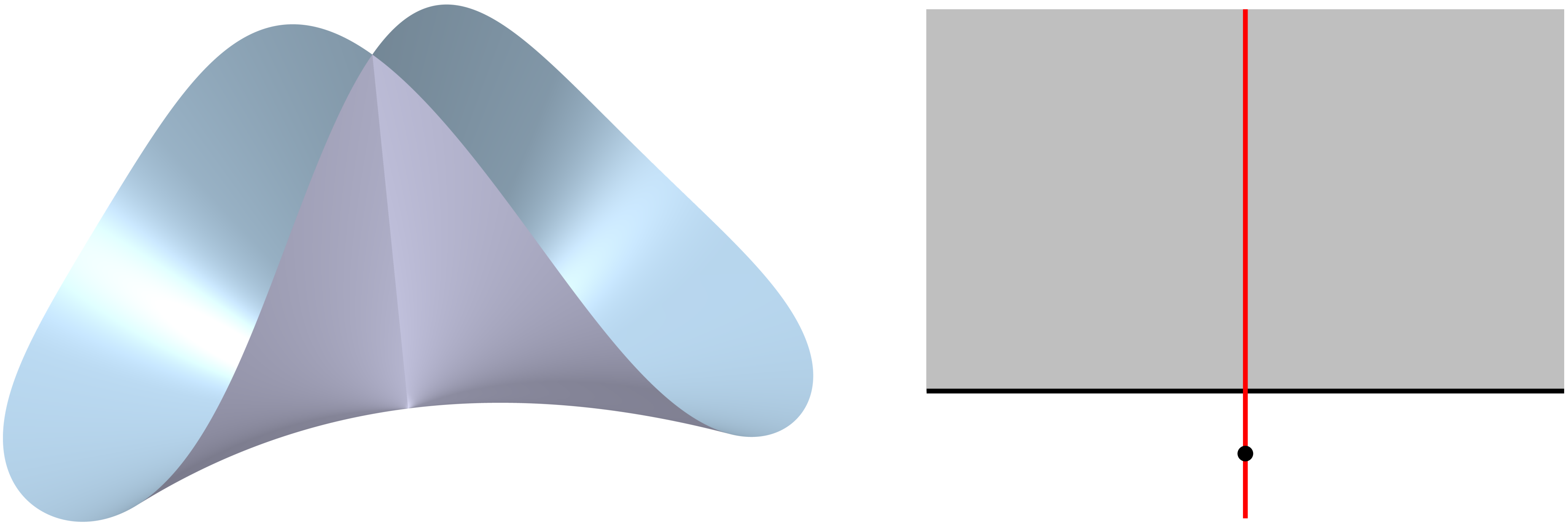
Problem. Which sets $X \subset \Omega$ have the analytic extension property?

Necessary condition. X is the zero set of an analytic function on $\Omega \rightsquigarrow X$ is a C-analytic set.

Example. The necessary condition is not sufficient. Consider Whitney's umbrella $W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3$ and

$$f : W \to \mathbb{R},\ (x, y, z) \mapsto \begin{cases} \frac{x}{z+1} & \text{if } (x, y, z) \neq (0, 0, -1), \\ 0 & \text{otherwise} \end{cases}$$

is analytic on W, but does not extend analytically to $\mathbb{R}^3$.



$$W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3 \quad \textsf{and} \quad \xi := \frac{x}{z+1}$$

A sufficient condition is provided by coherence and Cartan's Theorem B.

Coherence. A C-analytic set X is coherent if its local equations at each point $x \in X$ are generated by its global equations.

$$\mathcal{J}_{X,x} := \{f_x \in \mathcal{O}_{\mathbb{R}^n,x} : \ X_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{I}(X) := \{f \in \mathcal{O}(\mathbb{R}^n) : \ X \subset \mathcal{Z}(f)\}$$

$$X \text{ is coherent } \iff \mathcal{J}_{X,x} = \mathcal{I}_{X,x} := \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x} \ \forall x \in X$$

Cartan's Theorem B (1957) $\Longrightarrow$ If $X \subset \Omega$ is a coherent C-analytic set, X has the analytic extension property.

Theorem. *A set $X \subset \Omega$ has the analytic extension property $\iff$ X is a coherent analytic set.*

1.2. Nash case

Semialgebraic set: Boolean combination of sets defined by polynomial equalities and inequalities.

Semialgebraic function: Function with semialgebraic graph.

Nash function on an open semialgebraic set: Analytic + semialgebraic function on an open semialgebraic set.

Nash manifold: smooth manifold + semialgebraic set $\iff$ analytic manifold + semialgebraic set.

Nash set: Zero set of a Nash function on an open semialgebraic set $\iff$ C-analytic set + semialgebraic set.

Local Nash function on a Nash set: function on a Nash set that is the restriction of a Nash function on an open neighborhood of each point.

Nash extension property: Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set. A subset $X \subset \Omega$ has the *Nash extension property* if each local Nash function $f : X \to \mathbb{R}$ extends to a Nash function defined on Ω.

Problem. Which sets $X \subset \Omega$ have the Nash extension property?

Necessary condition: X is a Nash set.

Example. The necessary condition is not sufficient. Consider Whitney's umbrella $W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3$ and

$$f : W \to \mathbb{R},\ (x, y, z) \mapsto \begin{cases} \frac{x}{z+1} & \text{if } (x, y, z) \neq (0, 0, -1), \\ 0 & \text{otherwise} \end{cases}$$

is local Nash on W, but does not extend to $\mathbb{R}^3$ as a Nash function.

A sufficient condition is provided by coherence and Nash Theorem B.

Coherence: A Nash set X is coherent if its local equations at each point $x \in X$ are generated by its global equations.

$$\mathcal{J}^\bullet_{X,x} := \{f_x \in \mathcal{N}_{\mathbb{R}^n,x} : \ X_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{I}(X)^\bullet := \{f \in \mathcal{N}(\mathbb{R}^n) : \ X \subset \mathcal{Z}(f)\}$$

$$X \text{ is coherent } \iff \mathcal{J}^\bullet_{X,x} = \mathcal{I}^\bullet_{X,x} := \mathcal{I}^\bullet(X)\mathcal{N}_{\mathbb{R}^n,x} \ \forall x \in X$$

Nash Theorem B (Coste-Ruiz-Shiota, 2000) $\Longrightarrow$ If $X \subset \Omega$ is a coherent Nash set, X has the Nash extension property.

Theorem. $X \subset \Omega$ *has the Nash extension property* $\iff$ X *is a coherent Nash set.*

Remark. We 'semialgebraically' adapt the constructions done in the analytic case avoiding cohomology arguments.

Bad 'cohomological' behavior in the Nash case: $H^1(\mathbb{R}, \mathcal{N}_{\mathbb{R}}) \neq 0$ (Hubbard, 1972).

1.3. Related problem: Whitney's extension problem

Both previous problems are somehow related to Whitney's extension problem (1934) both in the general and the semialgebraic setting. Some relevant names:

$\mathcal{C}^p$ **case** (solved): Whitney, Glaeser, Bierstone, Milman, Pawłucki, Fefferman,. . . .

$\mathcal{C}^p$ **semialgebraic case** (not completely solved): Kurdyka, Pawłucki, Aschenbrenner, Thamrongthanyalak, Fefferman, Luli, Bierstone, Campesato, Milman,. . .

2. Analytic case

2.1. Coherence and Cartan's Theorems A & B

Coherence. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{R}^n}$-modules is *coherent* if:

(i) $\mathcal{F}$ is of finite type: $\forall x \in \mathbb{R}^n$ $\exists$ an open neighborhood $U \subset \mathbb{R}^n$ of x, $m \in \mathbb{N}^*$ and a surjective morphism $\mathcal{O}^m_{\mathbb{R}^n}|_U \to \mathcal{F}|_U$, and

(ii) the kernel of each homomorphism $\mathcal{O}^p_{\mathbb{R}^n}|_V \to \mathcal{F}|_V$ is of finite type for each $p \geq 1$ and each open subset V of $\mathbb{R}^n$.

Cartan's Theorems A and B. describe the local-global behavior of coherent sheaves $\mathcal{F}$ of $\mathcal{O}_{\mathbb{R}^n}$-modules:

(A) *The stalks of a coherent sheaf $\mathcal{F}$ are spanned by the global sections.*

(B) *Each p-cohomology group of a coherent sheaf $\mathcal{F}$ is trivial for each $p > 0$.*

Let $X \subset \mathbb{R}^n$ be a C-analytic subset:

$\mathcal{C}^\omega_X := \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X$ is the *sheaf of analytic functions germs on* X

$\mathcal{O}_X := \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X$ is the *sheaf of global analytic functions germs on* X.

$\mathcal{I}_X$ *is the biggest coherent* $\mathcal{O}_{\mathbb{R}^n}$*-sheaf of ideals with support* $X \Longrightarrow \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X$ is coherent.

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X \to 0 \quad \text{(exact sequence coherent sheaves)}$$

Cartan's Theorem B $\Longrightarrow H^1(\mathbb{R}^n, \mathcal{I}_X) = 0 \Longrightarrow$ The sequence

$$0 \to H^0(\mathbb{R}^n, \mathcal{I}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X) \to 0$$

is exact.

$\mathcal{O}(X) := H^0(X, (\mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X)|_X)$ is the *ring of global analytic functions on* X

$\mathcal{C}^\omega(X) := H^0(X, (\mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X)|_X)$ is the *ring of analytic functions on* X.

Analytic extension property: A C-analytic set $X \subset \mathbb{R}^n$ has the *analytic extension property* if $\mathcal{O}(\mathbb{R}^n) \to \mathcal{C}^\omega(X)$ is surjective.

2.2. Tails and points of non-coherence of a C-analytic set

Let $X \subset \mathbb{R}^n$ be a C-analytic set.

2.2.1. Complexification of a C-analytic set.

Consider the coherent sheaf of $\mathcal{O}_{\mathbb{C}^n}$-ideals $\mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathbb{R}^n$. There exists an open neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$ and a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{C}^n}$-ideals on Ω such that $\mathcal{F}|_{\mathbb{R}^n} = \mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{I}(X)\mathcal{O}_{\mathbb{C}^n}|_{\mathbb{R}^n}$. A *complexification* $\widetilde{X}$ of X is the support of $\mathcal{F}$.

2.2.2. Regular and singular points in the analytic setting

$x \in X$ is a *regular point of* X if $\mathcal{O}(X)_{\mathfrak{m}_x}$ is a regular local ring. If one between $\mathcal{O}(X)_{\mathfrak{m}_x}$, $\mathcal{O}_{\mathbb{R}^n,x}/\mathcal{I}_{X,x}$, $\mathcal{O}_{\mathbb{C}^n,x}/(\mathcal{I}_{X,x} \otimes_{\mathbb{R}} \mathbb{C})$, $\mathcal{O}(\widetilde{X})_{\mathfrak{n}_x}$ is regular, all are regular.

$\operatorname{Reg}(X) = \operatorname{Reg}(\widetilde{X}) \cap X \rightsquigarrow$ *set of regular points* of X.

$\operatorname{Sing}(X) := X \setminus \operatorname{Reg}(X) = \operatorname{Sing}(\widetilde{X}) \cap X \rightsquigarrow$ *singular locus of* X is a C-analytic set and $\dim(\operatorname{Sing}(X)) < \dim(X)$ (because $\dim(\operatorname{Sing}(\widetilde{X})) < \dim(\widetilde{X})$).

2.2.3. C-semianalytic sets

A C-semianalytic subset S of $\mathbb{R}^n$ is a locally finite union of *basic C-semianalytic subsets* of $\mathbb{R}^n$ $\rightsquigarrow$ $\{f = 0, g_1 > 0, \dots, g_r > 0\}$ where $r \geq 1$ and $f, g_i \in \mathcal{O}(\mathbb{R}^n)$.

2.2.4. Set of 'tails' of a C-analytic set

$T(X) := \{x \in X : \ \mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}\}$ is the set of 'tails' of X $\rightsquigarrow$ $T(X) \subset \operatorname{Sing}(X)$ is a C-semianalytic set of dimension $\dim(T(X)) < \dim(X)$.

2.2.5. Set of points of non-coherence.

The set $N(X)$ of points of non-coherence of X is the set of points $x \in X$ such that $\mathcal{J}_X$ is not of finite type at x (for each $x \in U \overset{\text{open}}{\subset} X$ the restricted sheaf $\mathcal{J}_X|_U$ is not of finite type) $\rightsquigarrow$ $N(X)$ is a closed C-semianalytic subset of X of dimension $\leq \dim(X) - 2$.

2.2.6. Properties of 'tails' and points of non-coherence

Let $X \subset \mathbb{R}^n$ be a C-analytic set.

(1) $\forall x \in N(X)$ $\exists$ an analytic arc $\alpha : (-1,1) \to X$ such that $\alpha(0) = x$, $\alpha((0,1)) \subset T(X) \setminus N(X)$.

(2) $\mathrm{Cl}(T(X)) = \mathrm{Cl}(T(X) \setminus N(X)) = T(X) \cup N(X)$.

(3) X is coherent $\iff T(X) = \varnothing \iff N(X) = \varnothing$

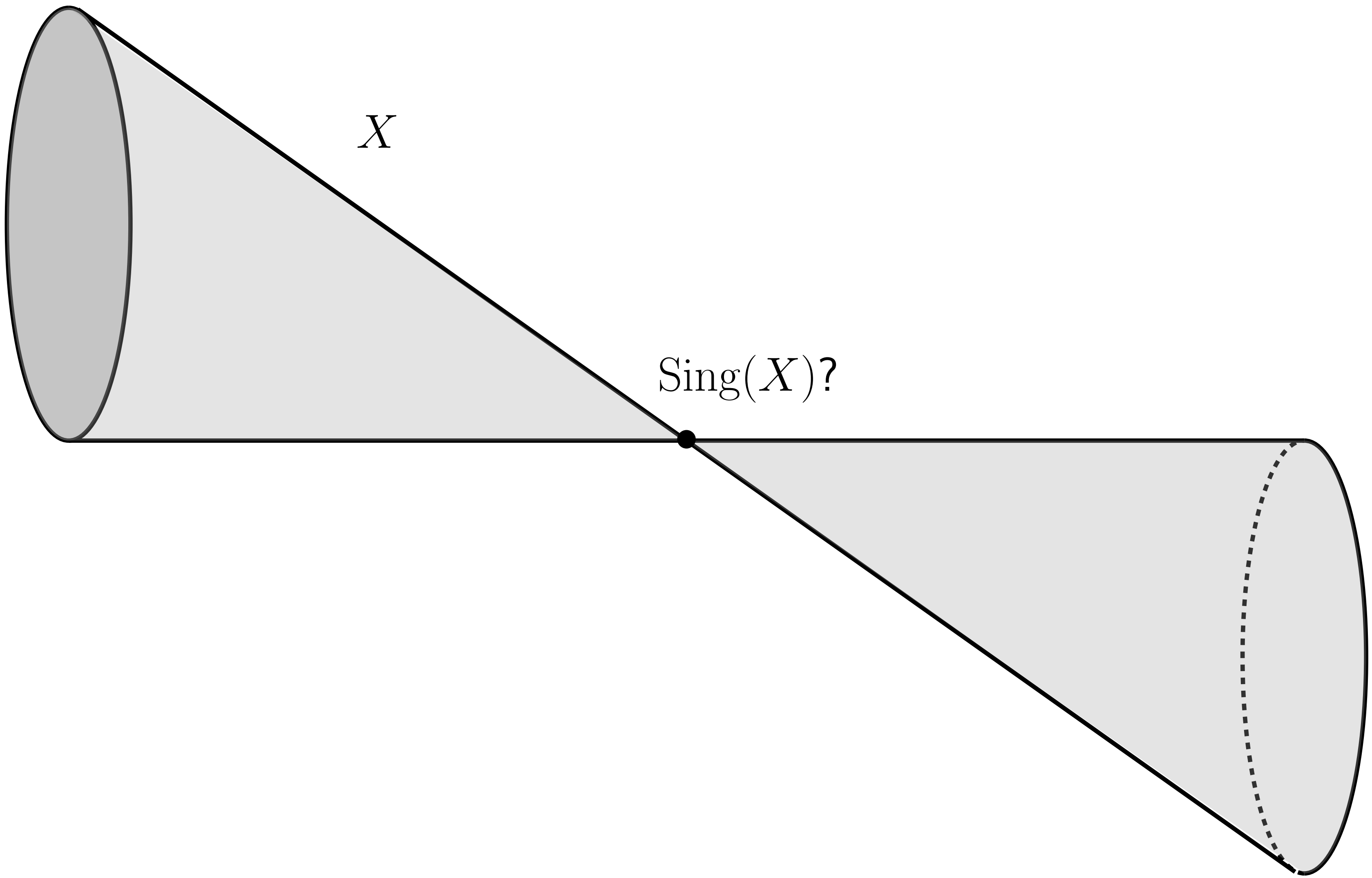
(4) If S is a connected component of $\mathrm{Cl}(T(X))$, then $S \cap N(X) \neq \varnothing$.

(5) $\dim(N(X)_x) < \dim(T(X)_x) \leq \dim(\mathrm{Sing}(X)_x)$ for each $x \in N(X)$.

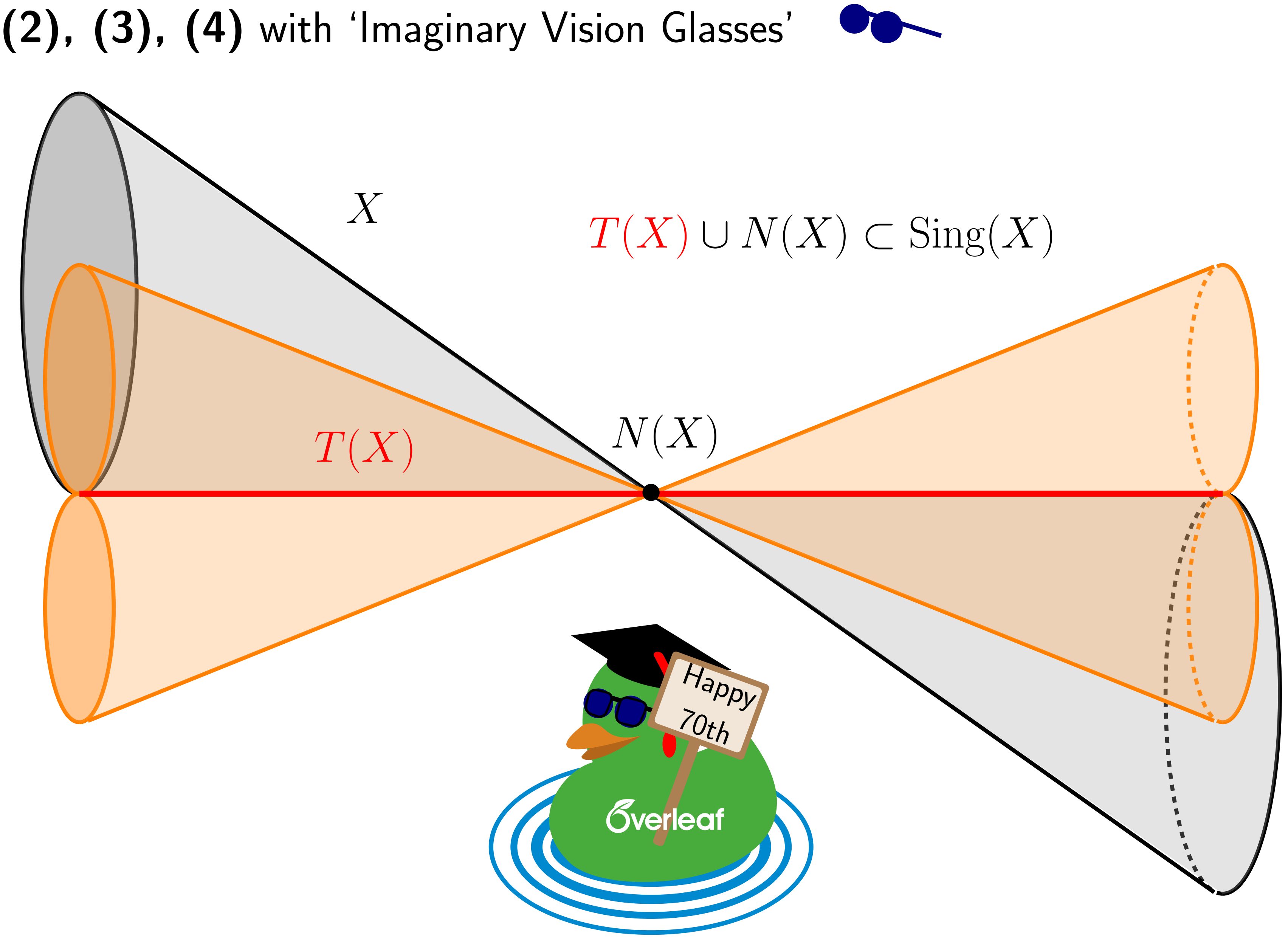
(6) $N(X) \cap T(X)$ may be non-empty, even if X is C-irreducible. Consider $X := \{(x^2-y^2-x^3)^2(y-x)-z^2 = 0\} \rightsquigarrow T(X) = \{x^2-y^2-x^3 = 0, x-y > 0\} \cup \{(0,0,0)\}$ and $N(X) = \{(0,0,0)\}$.

(7) A general idea in Real Geometry is that non-coherence arises when the irreducible components of the objects are not pure dimensional.

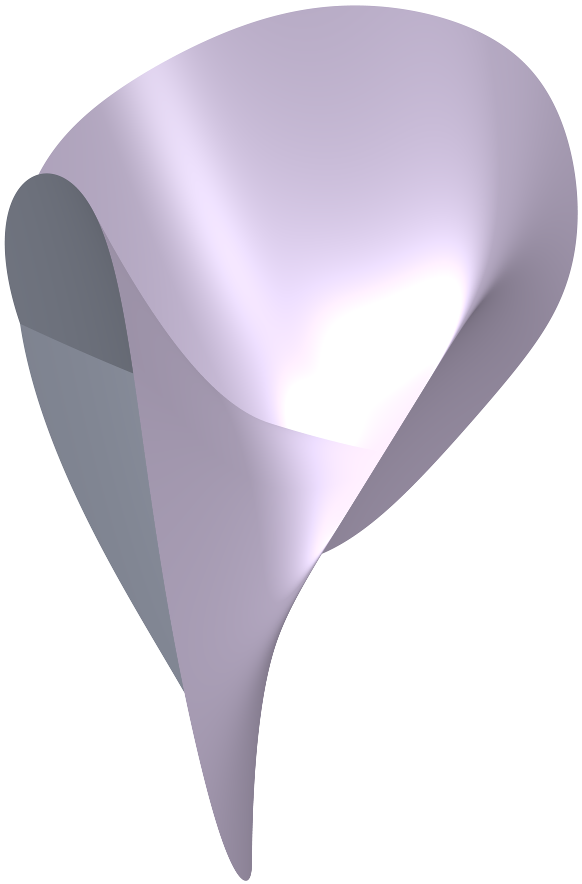
(1), (2), (3), (4) with 'Real Vision Glasses'

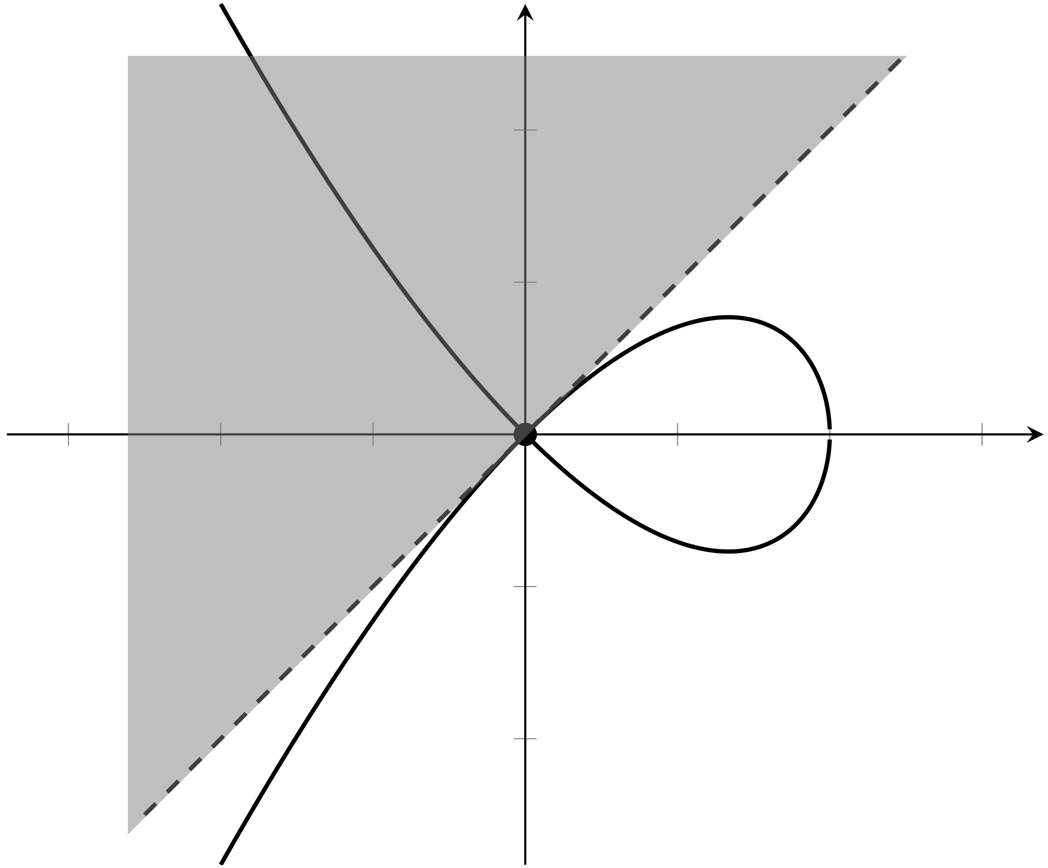


(1), (2), (3), (4) with 'Imaginary Vision Glasses'



(6)





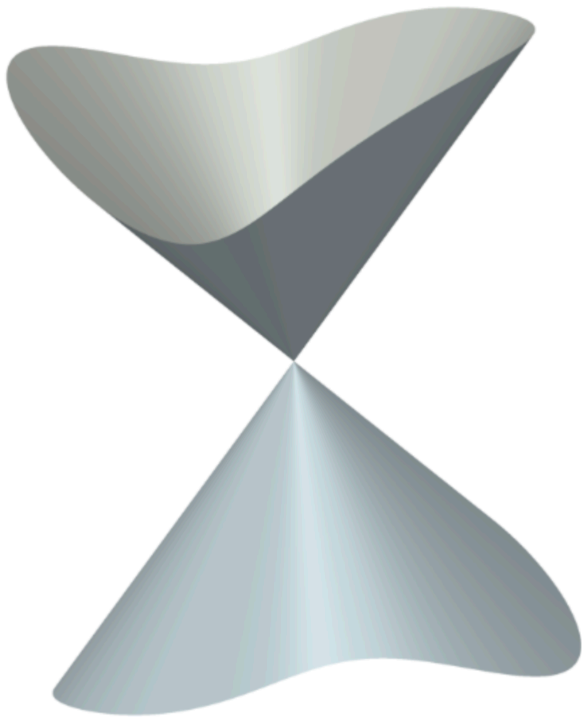
$$X := \{(x^2 - y^2 - x^3)^2(y - x) - z^2 = 0\}$$

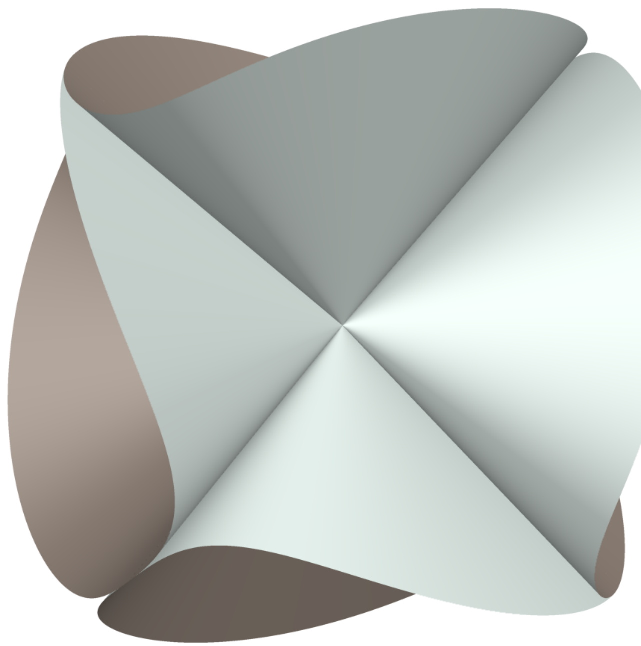
$$N(X) \cap T(X) = \{(0,0,0)\} \neq \varnothing$$

2.3. Examples of pure dimensional non-coherent C-analytic sets

(i) Galbiati-Hironaka: $X := \{z(x+y)(x^2+y^2) - x^4 = 0\} \subset \mathbb{R}^3 \rightsquigarrow N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \operatorname{Sing}(X) \setminus N(X)$.

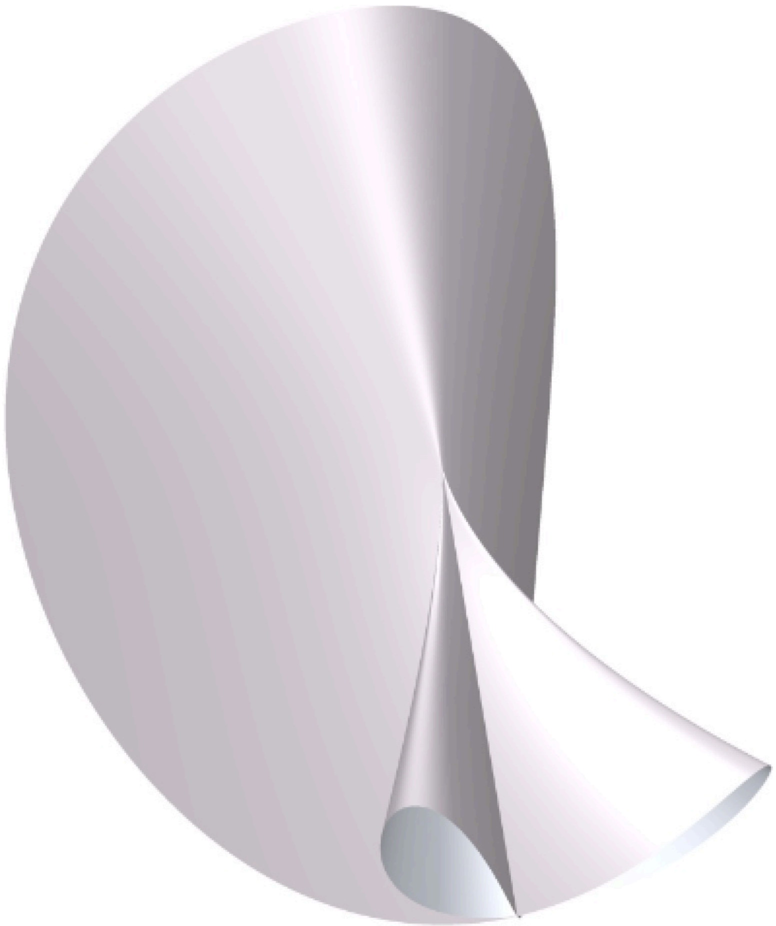
(ii) Galbiati-Hironaka (modified): $X := \{z^2(x+y)^2(x^2+y^2) - x^6 = 0\} \subset \mathbb{R}^3$ $\rightsquigarrow N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, yz = 0\}$ and $T(X) = \{x = 0, y = 0\} \setminus N(X)$.

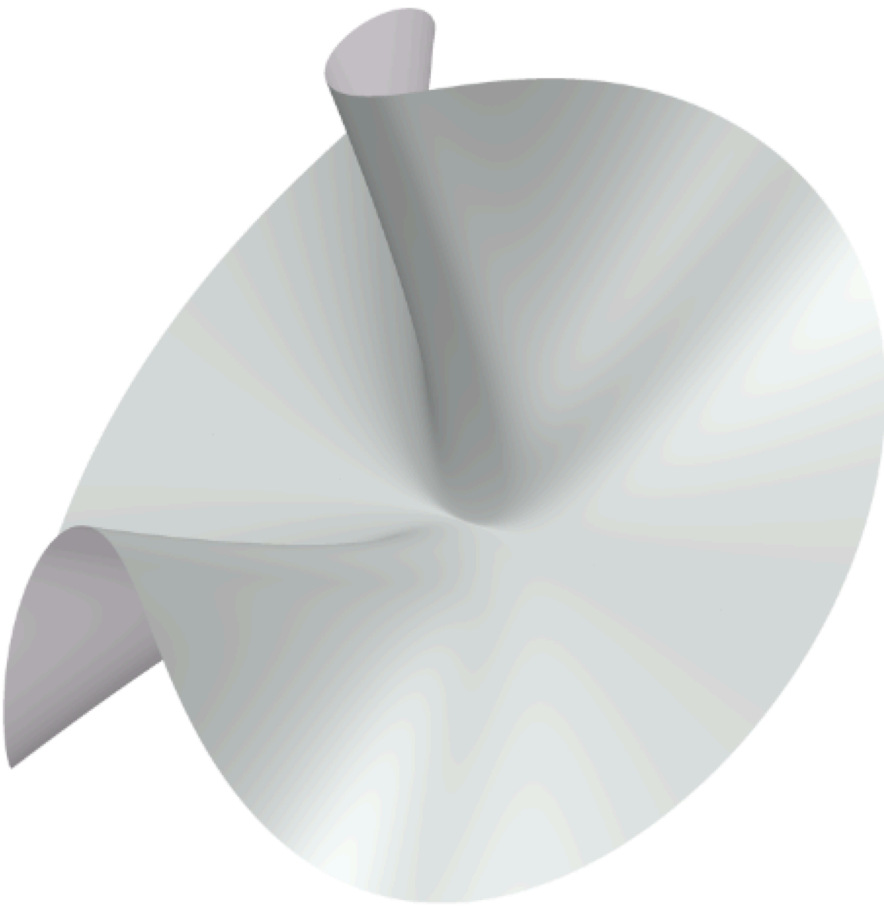




(iii) Birdie non-coherent singularity: $X := \{(x^2 + zy^2)x - y^4 = 0\} \subset \mathbb{R}^3 \rightsquigarrow N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \operatorname{Sing}(X) \cap \{z < 0\}$.

(iv) Fake blanket: $X := \{(x^2 + z^2y^2)x - y^4 = 0\} \subset \mathbb{R}^3 \rightsquigarrow N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \operatorname{Sing}(X) \setminus N(X)$.





2.4. Obstructing set of a meromorphic function

(1) Let X be a C-analytic subset of $\mathbb{R}^n$, let $\zeta : X \to \mathbb{R}$ and $x \in X$ such that $\exists f_x, g_x \in \mathcal{O}_{\mathbb{R}^n,x}$ satisfying $\zeta_x = \frac{f_x}{g_x}$ and g_x does not belong to a minimal prime of $\mathcal{J}_{X,x}$.

$$\frac{f_x}{g_x} = -a_x \in \mathcal{O}_{\mathbb{R}^n,x} \iff f_x + a_x g_x \in \mathcal{J}_{X,x} \iff f_x \in g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{J}_{X,x}.$$

(2) Let $\{X_i\}_{i\in I}$ be the C-analytic irreducible components of the C-analytic set X and let $\zeta := \frac{f}{g} \in \mathcal{M}(X)$ such that $f, g \in \mathcal{O}(\mathbb{R}^n)$ and $g|_{X_i} \neq 0$ for each $i \in I$.

$$\zeta = \frac{f}{g} = -a|_X \text{ where } a \in \mathcal{O}(\mathbb{R}^n) \iff f \in g\mathcal{O}(\mathbb{R}^n) + \mathcal{I}(X)$$
$$\implies f_x \in g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x} = g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x} \ \forall x \in X$$

Obstructing set of $\zeta = \frac{f}{g} \in \mathcal{M}(X)$: $\mathsf{O}(\zeta) := \{x \in X : \ f_x \notin g_x\mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x}\}$ (closed subset of X).

Remark. $\zeta \in \mathcal{M}(X)$ has an analytic extension to $\mathbb{R}^n \iff \mathsf{O}(\zeta) = \varnothing$.

2.5. Main Theorem

Theorem. *Let* $X \subset \mathbb{R}^n$ *be a* C*-analytic set with* $N(X) \neq \varnothing$*. Let*

- $Y \subset X$ *be a* C*-analytic subset that contains no irreducible component of* X *and meets* $T(X)$*,*
- $U_0 \subset \mathbb{R}^n$ *an open neighborhood of* Y*,*
- $h \in H^0(U_0, \mathcal{J}_X)$ *such that* $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ *for each* $y \in Y \cap T(X)$*.*

$\exists \zeta \in (\mathcal{M}(X) \cap \mathcal{C}^\omega(X)) \setminus \mathcal{O}(\mathbb{R}^n)$ *and* $Y \cap T(X) \subset \mathsf{O}(\zeta) \subset Y \cap \mathrm{Cl}(T(X))$*.*

2.6. Winning family of denominators

Let $Y \subset \mathbb{R}^n$ be a C-analytic subset of $\mathbb{R}^n$ and let $\widetilde{Y} \subset \Omega$ be an invariant Stein complexification of Y closed in an open neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$. Write $\widetilde{Y} = \mathcal{Z}(P_1, \ldots, P_m)$ for some invariant $P_1, \ldots, P_m \in \mathcal{O}(\Omega)$.

Define

$$P_\lambda := \lambda_1 P_1^2 + \cdots + \lambda_m P_m^2 \in \mathcal{O}(\Omega)$$

where $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathfrak{Q}_m := \{\lambda_1 > 0, \ldots, \lambda_m > 0\} \subset \mathbb{R}^m$. If $\Omega \subset \mathbb{C}^n$ is a contractible invariant open Stein neighborhood of $\mathbb{R}^n$, we can find a square-free invariant $P_\lambda^* \in \mathcal{O}(\Omega)$ such that $P_\lambda^* | P_\lambda$ and $\mathcal{Z}(P_\lambda^*) = \mathcal{Z}(P_\lambda)$.

Examples. (i) $Y := \{q\}$, $\widetilde{Y} := \{q\}$ and $P_i := \mathtt{x}_i - q_i$ for $i = 1, \ldots, n$.

(ii) $Y := \{y_k\}_{k \geq 1}$ is a discrete subset of $T(X)$ and $\widetilde{Y} = Y$. If $Y = \mathbb{Z} \times \{(0, \overset{(n-1)}{\ldots}, 0)\}$, we take $P_1(x) := \sin(2\pi x_1)$, $P_2(x) = x_2$, $\ldots$, $P_n(x) = x_n$.

2.7. Special global equations outside $N(X)$ ('Almost numerators')

Theorem. *Let $X \subset \mathbb{R}^n$ be a C-analytic set. Then there exists $h \in \mathcal{I}(X \setminus N(X))$ such that $\mathcal{Z}(h) = X \setminus N(X)$ and $h_x \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ for each $x \in T(X) \setminus N(X)$.*

2.7.1. C-analytic and Nash locally hypersurfaces

A C-analytic set $X \subset \mathbb{R}^n$ is a *C-analytic locally hypersurface* if for each $x \in \mathbb{R}^n$ the ideal $\mathcal{J}_{X,x}$ of $\mathcal{O}_{\mathbb{R}^n,x}$ is principal (in particular pure dimensional).

Remark. Not every C-analytic hypersurface is a C-analytic locally hypersurface.

Lemma. *Let $X \subset \mathbb{R}^n$ be a C-analytic locally hypersurface. Then there exists $h \in \mathcal{O}(\mathbb{R}^n \setminus N(X))$ such that h_x generates the ideal $\mathcal{J}_{X,x}$ of $\mathcal{O}_{\mathbb{R}^n,x}$ for each $x \in \mathbb{R}^n \setminus N(X)$.*

2.7.2. Winning equations around a C-analytic subset ('Numerators')

Let $X \subset \mathbb{R}^n$ be a C-analytic set and let $Y \subset X$ be a C-analytic subset.

(1) Let $h \in \mathcal{O}(\mathbb{R}^n \setminus N(X))$ be such that $\mathcal{Z}(h) = X \setminus N(X)$ and $h_x \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ $\forall x \in T(X) \setminus N(X)$.

If $Y \subset X \setminus N(X)$, then $U_0 := \mathbb{R}^n \setminus N(X)$ is an open neighborhood of Y and $h \in \mathcal{O}(U_0)$ satisfies $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ $\forall x \in Y \cap T(X)$.

(2) If $Y = \{y_k\}_{k\geq 1}$ is a discrete set, $\forall y_k \in Y \cap T(X)$ $\exists h_{k,y_k} \in \mathcal{J}_{X,y_k} \setminus \mathcal{I}_{X,y_k}$ such that $\mathcal{Z}(h_{k,y_k}) = X_{y_k}$. $\forall y_k \in Y \setminus T(X)$ let $h_{k,y_k} \in \mathcal{I}_{X,y_k}$ be such that $\mathcal{Z}(h_{k,y_k}) = X_{y_k}$.

Let $\{V_k\}_{k\geq 1}$ be pairwise disjoint open neighborhoods in $\mathbb{R}^n$ of the point y_k and h_k an analytic representative of h_{k,y_k} in V_k $\forall k \geq 1$. Define $U_0 := \bigsqcup_{k\geq 1} V_k$ and

$$h : U_0 \to \mathbb{R},\ x \mapsto h_k(x) \text{ if } x \in V_k.$$

$h \in \mathcal{O}(U_0)$ satisfies $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ $\forall x \in Y \cap T(X)$.

No restriction with respect to the set $N(X)$.

2.8. Proof of the Main Theorem

STEP 1. *Initial preparation.* Consider the exact sequence of sheaves

$$0 \to \mathcal{J}_X \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X \to 0$$

and the exact long corresponding sequence of global sections

$$\begin{aligned} 0 \to H^0(\mathbb{R}^n, \mathcal{J}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \to H^1(\mathbb{R}^n, \mathcal{J}_X) \to H^1(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \\ \to \cdots \to H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \to H^{p+1}(\mathbb{R}^n, \mathcal{J}_X) \to H^{p+1}(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}). \end{aligned}$$

Cartan's theorem B $\Longrightarrow H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) = 0 \ \forall p \geq 1 \Longrightarrow$

$$0 \to H^0(\mathbb{R}^n, \mathcal{J}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \xrightarrow{\delta} H^1(\mathbb{R}^n, \mathcal{J}_X) \to 0.$$

$H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X)$ is surjective $\iff H^1(\mathbb{R}^n, \mathcal{J}_X) = 0.$

If X is coherent $\overset{\text{(Cartan's B)}}{\Longrightarrow} H^1(\mathbb{R}^n, \mathcal{J}_X) = 0.$

Purpose: X *is non-coherent* $\Longrightarrow H^1(\mathbb{R}^n, \mathcal{J}_X) \neq 0.$

2.8.1. Relations for 1-cocycle and 1-coboundary

Let $\mathfrak{U} := \{U_0, U_1\}$ be an open covering of $\mathbb{R}^n$. Let $\mathcal{F}$ be a sheaf on $\mathbb{R}^n$.

Pick $i, j, k \in \{0, 1\}$ and $f_{ij} \in \mathcal{F}(U_i \cap U_j), f_{ik} \in \mathcal{F}(U_i \cap U_k), f_{jk} \in \mathcal{F}(U_j \cap U_k)$.

1-**cocycle relations**: $f_{00} = 0$ on U_0, $f_{11} = 0$ on U_1 and $f_{10} = -f_{01}$ on $U_0 \cap U_1$.

1-**coboundary relations** for $f_0 \in \mathcal{F}(U_0)$ and $f_1 \in \mathcal{F}(U_1)$:

$$\begin{aligned} U_0 \cap U_1 &\rightsquigarrow f_{01} = f_1 - f_0, \\ U_1 \cap U_0 &\rightsquigarrow f_{10} = f_0 - f_1. \end{aligned}$$

A 1-**cocycle**

$$(f_{00}, f_{01}, f_{10}, f_{11}) = (0, f_{01}, -f_{01}, 0) \in \mathcal{F}(U_0 \cap U_0) \times \mathcal{F}(U_0 \cap U_1) \times \mathcal{F}(U_1 \cap U_0) \times \mathcal{F}(U_1 \cap U_1)$$

is a 1-**coboundary** $\iff \exists f_i \in \mathcal{F}(U_i)$ for $i \in \{0, 1\}$ such that $f_{01} = f_1 - f_0$.

Step 2. *Construction of an open covering* $\mathfrak{U} := \{U_0, U_1\}$ *of* X *and a* 1*-cocycle* $f_{01} \in H^0(U_0 \cap U_1, \mathcal{J}_X)$ *that is not a* 1*-coboundary.*

By hypothesis we have a C-analytic set $Y \subset X$, $Y \subset U_0 \overset{\text{open}}{\subset} \mathbb{R}^n$ and $h \in \mathcal{O}(U_0)$ such that $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ $\forall y \in Y \cap T(X)$.

Let $\widetilde{Y} \subset \Omega \overset{\text{open}}{\subset} \mathbb{C}^n$ be a complexification of Y and let $P_1, \ldots, P_m \in \mathcal{O}(\Omega)$ be invariant such that $\widetilde{Y} = \mathcal{Z}(P_1, \ldots, P_m)$.

$\mathfrak{U} := \{U_0, U_1 := \mathbb{R}^n \setminus Y\}$ open covering of $\mathbb{R}^n$. Define

$$P_\lambda(\mathsf{z}) := \lambda_1 P_1^2 + \cdots + \lambda_m P_m^2 \quad \text{and} \quad \boxed{f_{01,\lambda}(\mathsf{z}) := \frac{h(\mathsf{z})}{P_\lambda(\mathsf{z})} \in H^0(U_0 \cap U_1 = U_0 \setminus Y, \mathcal{J}_X)}$$

where $\lambda \in \mathfrak{Q}_m := \{\lambda_1 > 0, \ldots, \lambda_m > 0\}$.

Let $f_0 \in H^0(U_0, \mathcal{J}_X)$ and $f_1 \in H^0(U_1, \mathcal{J}_X)$ be such that $f_{01,\lambda} = f_1 - f_0$ on $U_0 \cap U_1$.

$$g_\lambda := f_1 P_\lambda = h + f_0 P_\lambda \in \mathcal{O}(\mathbb{R}^n) \quad \text{and} \quad g_{k,\lambda}|_X = 0$$

If G_λ is an analytic extension to $\Omega' \subset \Omega \Longrightarrow G_\lambda|_{\widetilde{X} \cap \Omega'} = 0$.

Let $V_0 \subset \mathbb{C}^n$ be an open neighborhood of U_0 such that h, f_0 extend to invariant holomorphic functions H, F_0 on V_0:

$$G_\lambda|_{V_0} = H + F_0 P_\lambda|_{V_0}.$$

We choose $\lambda_0 \in \mathfrak{Q}_m$ such that $\dim_{\mathbb{C}}((\widetilde{X}_y \cap \mathcal{Z}(P_{\lambda_0,y}) \setminus \mathcal{Z}(H_y)) \geq 1 \ \forall y \in Y \cap T(X)$.

$\forall y \in Y \cap T(X) \ \exists \beta^y : (-1,1) \to \widetilde{X}$ analytic curve such that $\beta^y(0) = y$ and $\beta^y((0,1)) \subset (\widetilde{X} \cap \mathcal{Z}(P_{\lambda_0})) \setminus \mathcal{Z}(H)$. Then

$$G_{\lambda_0} \circ \beta^y = 0, P_{\lambda_0} \circ \beta^y = 0, H \circ \beta^y \neq 0$$
$$\rightsquigarrow 0 = G_{\lambda_0} \circ \beta^y = H \circ \beta^y + (F_0 \circ \beta^y)(P_{\lambda_0} \circ \beta^y) = H \circ \beta^y \quad !!!!!$$

Conclusion: $f_{01,\lambda_0} \in H^0(U_0 \cap U_1, \mathcal{J}_X)$ is 1-cocycle that it is not a 1-coboundary.

The obstruction concentrates at all the points of $Y \cap T(X)$.

Step 3. *Construction of* $\zeta \in \mathcal{C}^\omega(X) \setminus \mathcal{O}(X)$. (We use the 1-cocycle)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(U_0, \mathcal{J}_X) \times H^0(U_1, \mathcal{J}_X) & \longrightarrow & \mathcal{O}_{\mathbb{R}^n}(U_0) \times \mathcal{O}_{\mathbb{R}^n}(U_1) & \longrightarrow & H^0(U_0, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \times H^0(U_1, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(U_0 \cap U_1, \mathcal{J}_X) & \longrightarrow & \mathcal{O}_{\mathbb{R}^n}(U_0 \cap U_1) & \longrightarrow & H^0(U_0 \cap U_1, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
(f_0, f_1) & \longmapsto & (f_0, f_1) \\
\downarrow & & \downarrow \\
f_1 - f_0 & \longmapsto & f_1 - f_0
\end{array}$$

$$\begin{array}{ccc}
(q_0, q_1) & \longmapsto & ([q_0], [q_1]) \\
\downarrow & & \downarrow \\
q_1 - q_0 & \longmapsto & [q_1 - q_0]
\end{array}$$

We seek $(q_0, q_1) \in \mathcal{O}_{\mathbb{R}^n}(U_0) \times \mathcal{O}_{\mathbb{R}^n}(U_1)$ such that

$$\frac{h}{P_{\lambda_0}} = f_{01,\lambda_0} = q_1 - q_0 = \delta((q_0, q_1))$$

$$\rightsquigarrow \; h_{\lambda_0} := q_1 P_{\lambda_0} = h + q_0 P_{\lambda_0} \in \mathcal{O}(\mathbb{R}^n) \quad \& \quad q_0 := \frac{h_{\lambda_0} - h}{P_{\lambda_0}} \in \mathcal{O}(U_0).$$

Consider the exact sequence of coherent $\mathcal{O}_{\mathbb{R}^n}$-sheaves

$$0 \to P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n} \to 0.$$

Cartan's Theorem B $\Longrightarrow$

$$0 \to H^0(\mathbb{R}^n, P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n}) \to 0 \quad \text{(exact)}$$

As $h \in H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n})$, $\exists h_{\lambda_0} \in \mathcal{O}(\mathbb{R}^n)$ such that $h_{\lambda_0} - h$ is a multiple of P_{λ_0}.

$$\begin{cases} q_0 := \frac{h_{\lambda_0}-h}{P_{\lambda_0}} \in \mathcal{O}(U_0), \\ q_1 := \frac{h_{\lambda_0}}{P_{\lambda_0}} \in \mathcal{O}(U_1) \qquad \text{(because } \mathcal{Z}(P_{\lambda_0}) = Y\text{)} \end{cases} \quad \& \quad \delta((q_0, q_1)) = q_1 - q_0 = \frac{h}{P_{\lambda_0}} = f_{01}$$

Sought function.

$$q_{\lambda_0} : X \to \mathbb{R},\ z \mapsto \begin{cases} q_0(z) & \text{if } z \in X \cap U_0 = X \setminus N(X), \\ q_1(z) & \text{if } z \in X \cap U_1 = X \setminus Y \end{cases} \quad \& \quad \boxed{\xi_{\lambda_0} = \frac{h_{\lambda_0}}{P_{\lambda_0}} \in \mathcal{M}(X)}$$

Step 4. $Y \cap T(X) \subset \mathsf{O}(\xi_{\lambda_0}) \subset Y \cap \mathrm{Cl}(T(X))$.

3. Semialgebraic case

Let $S \subset \mathbb{R}^n$ be a semialgebraic set and denote $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$.

3.1. Differentiable semialgebraic functions

Definitions. *A* semialgebraic jet on S of order $p \geq 0$ *is a collection of semialgebraic functions* $F := (f_\alpha)_{|\alpha| \leq p}$ *on* S *(*$\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ *and* $|\alpha| := \alpha_1 + \cdots + \alpha_n$*). For each* $a \in S$ *write*

$$T_a^p F := \sum_{|\alpha| \leq p} \frac{f_\alpha(a)}{\alpha!} (\mathtt{x} - a)^\alpha \quad \textit{and} \quad R_a^p F := f_0 - T_a^p F.$$

$\mathcal{J}^p(S)$ *is the set of all semialgebraic jets on* S *of order* p*. Consider the linear map*

$$D^\beta : \mathcal{J}^p(S) \to \mathcal{J}^{p-|\beta|}(S), \ F := ((f_\alpha)_{|\alpha| \leq p}) \mapsto F_\beta := (f_{\gamma+\beta})_{|\gamma| \leq p-|\beta|},$$

$\forall \beta \in \mathbb{N}^n$ *with* $|\beta| \leq p$*.*

(Def.1) A continuous semialgebraic function $f : S \to \mathbb{R}$ is a $\mathcal{C}^p$ *semialgebraic function* if there exists a semialgebraic jet $F := (f_\alpha)_{|\alpha|\leq p}$ on S of order p such that $f_0 = f$ and for each β with $|\beta| \leq p$ and every point $a \in S$ it holds $|R_x^{p-|\beta|} F_\beta(y)| = o(\|x-y\|^{p-|\beta|})$ for $x, y \in S$ when $x, y \to a$.

Other alternatives to define $\mathcal{S}^p$-functions on a general semialgebraic set $S \subset \mathbb{R}^n$:

(Def.2) $\exists S \subset U \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and a $\mathcal{S}^p$-function $F : U \to \mathbb{R}$ such that $F|_S = f$.

(Def.3) $\forall x \in S\ \exists x \in U^x \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and $\mathcal{S}^p$-function $F_x : U^x \to \mathbb{R}$ such that $F_x|_{U^x\cap S} = f|_{U^x \cap S}$.

(Def.4) $\forall x \in S\ \exists x \in U^x \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and $\mathcal{C}^p$ function $F_x : U^x \to \mathbb{R}$ (non-necessarily semialgebraic) such that $F_x|_{U^x\cap S} = f|_{U^x\cap S}$.

(Def.5) $\exists S \subset U \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and a $\mathcal{C}^p$ function $F : U \to \mathbb{R}$ such that $F|_S = f$.

(Def.2) $\Longrightarrow$ **(Def.3)** $\Longrightarrow$ **(Def.4)** $\Longleftrightarrow$ **(Def.5)** and **(Def.2)** $\Longrightarrow$ **(Def.1)**.

3.1.1. Some relations

(1) (Def.4) $\Longrightarrow$ **(Def.5)**, using a $\mathcal{C}^p$-partition of unity.

(2) If $S \subset \mathbb{R}^n$ is compact, **(Def.3)** $\Longrightarrow$ **(Def.2)** using an $\mathcal{S}^p$-partition of unity.

(3) If $S \subset \mathbb{R}^n$ is closed, **(Def.1)** $\Longrightarrow$ **(Def.2)** (Kurdyka-Pawłucki, Thamrongthanyalak).

(4) If $n = 2$ and $S \subset \mathbb{R}^n$ is closed, **(Def.5)** $\Longrightarrow$ **(Def.2)** (Fefferman-Luli).

(5) If $p = 1$, $n \geq 1$ and $S \subset \mathbb{R}^n$ is closed, **(Def.5)** $\Longrightarrow$ **(Def.2)** (Aschenbrenner-Thamrongthanyalak).

(6) Weak version of '**(Def.5)** $\Longrightarrow$ **(Def.2)**' ($S \subset \mathbb{R}^n$ closed) via $t : \mathbb{N} \to \mathbb{N}$, which encodes *a certain loss of differentiability*: If there exist $S \subset U \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and a $\mathcal{C}^{t(p)}$ function on U such that $G|_S = f$, there exists an $\mathcal{S}^p$-function F on U such that $F|_S = f$ (Bierstone-Campesato-Milman).

(7) (Def.1) implies the existence of a $\mathcal{S}^{p-1}(U)$ function F on an open semialgebraic neighborhood $U \subset \mathbb{R}^n$ of S.

3.2. Smooth semialgebraic functions and Nash functions

Classical result: smooth + semialgebraic function on a Nash manifold $S \subset \mathbb{R}^n \iff$ analytic + algebraic function on S.

Question: What happens in the general semialgebraic setting?

The *ring of smooth semialgebraic functions on* S is

$$\mathcal{S}^{(\infty)}(S) := \bigcap_{p \geq 0} \mathcal{S}^p(S)$$

where $\mathcal{S}^p(S)$ is the ring of $\mathcal{S}^p$ functions on S.

Lemma. *If* $f \in \mathcal{S}^{(\infty)}(S)$, $\forall x \in S\ \exists F_x \in \mathcal{N}_{\mathbb{R}^n,x}$ *such that* $F_x|_{S_x} = f_x$.

Definition. $\mathcal{N}(S) := H^0(S, (\mathcal{N}_{\mathbb{R}^n})|_S) = \varinjlim \mathcal{N}(V)|_S$ *where* $V \subset \mathbb{R}^n$ *covers the open semialgebraic neighborhoods of* S. *We have* $\mathcal{N}(S) \subset \mathcal{S}^{(\infty)}(S)$.

Problem. For which semialgebraic sets $S \subset \mathbb{R}^n$ do we have $\mathcal{N}(S) = \mathcal{S}^{(\infty)}(S)$?

3.3. Nash sets

Let $X \subset \mathbb{R}^n$ be a Nash set.

(1) $\mathcal{S}^{(\infty)}(X) = \mathcal{C}^{\mathcal{N}}(X)$ and $\mathcal{N}(X) = H^0(X, \mathcal{N}_{\mathbb{R}^n}/\mathcal{I}^{\bullet}_X)$.

(2) $\mathcal{S}^{(\infty)}(X) = \mathcal{C}^{\mathcal{N}}(X) = \mathcal{N}(X) \iff X$ is coherent ($\mathcal{J}^{\bullet}_{X,x} = \mathcal{I}^{\bullet}_{X,x}\ \forall x \in X$).

3.4. Semialgebraic sets

Let $S \subset \mathbb{R}^n$ be a semialgebraic set and define

$$\mathcal{J}^{\bullet}_{S,x} := \{f_x \in \mathcal{N}_{\mathbb{R}^n,x} :\ S_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{C}^{\mathcal{N}}(S) := H^0(S, \mathcal{N}_{\mathbb{R}^n}/\mathcal{J}^{\bullet}_S).$$

3.4.1. Ring of smooth semialgebraic functions

Theorem. *Let $f : S \to \mathbb{R}$ be a function:*

$$f \in \mathcal{S}^{(\infty)}(S) \iff f \in \mathcal{C}^{\mathcal{N}}(S) \iff f \text{ is semialgebraic}$$
$$\text{and } \forall x \in S\ \exists F_x \in \mathcal{N}_{\mathbb{R}^n,x} \text{ such that } F_x|_{S_x} = f_x.$$

3.4.2. Ring of Nash functions

There exist $S \subset U \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and a Nash set $X \subset \mathbb{R}^n$ such that: if $S \subset V \overset{\text{open s.a.}}{\subset} U$ and $Y \subset V$ is the Nash closure of S in V, there exists $S \subset W \overset{\text{open s.a.}}{\subset} V$ of S such that $Y \cap W = X \cap W$.

$\mathcal{N}(S) = H^0(S, (\mathcal{N}_{\mathbb{R}^n}/\mathcal{I}_X^\bullet)|_S) = \varinjlim \mathcal{N}(V)/\mathcal{I}^\bullet(X)\mathcal{N}(V)$ where $V \subset \mathbb{R}^n$ covers the open semialgebraic neighborhoods of S.

3.4.3. Smooth semialgebraic functions versus Nash functions

We have $\mathcal{I}_{X,x}^\bullet \subset \mathcal{J}_{S,x}^\bullet$ for each $x \in S$. Define $A(S) := \{x \in S : \mathcal{I}_{X,x}^\bullet \neq \mathcal{J}_{S,x}^\bullet\}$.

$$\mathcal{N}(S) = \mathcal{S}^{(\infty)}(S) \iff A(S) = \varnothing$$

If $A(S) = \varnothing$, then $\mathcal{I}_{X,x}^\bullet = \mathcal{J}_{S,x}^\bullet$ $\forall x \in S$ and

$$\mathcal{S}^{(\infty)}(S) = \mathcal{C}^{\mathcal{N}}(S) = H^0(S, \mathcal{N}_{\mathbb{R}^n}/\mathcal{J}_S^\bullet) = H^0(S, (\mathcal{N}_{\mathbb{R}^n}/\mathcal{I}_X^\bullet)|_S) = \mathcal{N}(S).$$

Example. $S := \{x^2 - zy^2 = 0, z \geq 0\}$ *(Whitney's umbrella with the handle erased) has* $A(S) = \varnothing$.