Outer Lipschitz geometry of definable surface germs

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Tame Geometry and Extensions of Functions - Pawlucki 70

June 23-27, 2025

A set $X \subset \mathbb{R}^n$ inherits two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y.
X is **normally embedded** if these two metrics on X are equivalent.

A **surface germ** X is a closed two-dimensional germ at $\mathbf{0} \in \mathbb{R}^\mathbf{n}$, **definable** in a polynomially bounded o-minimal structure with the field of exponents $\mathbb{F}$. Surface germs X and Y are outer (inner) **Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $X \to Y$.

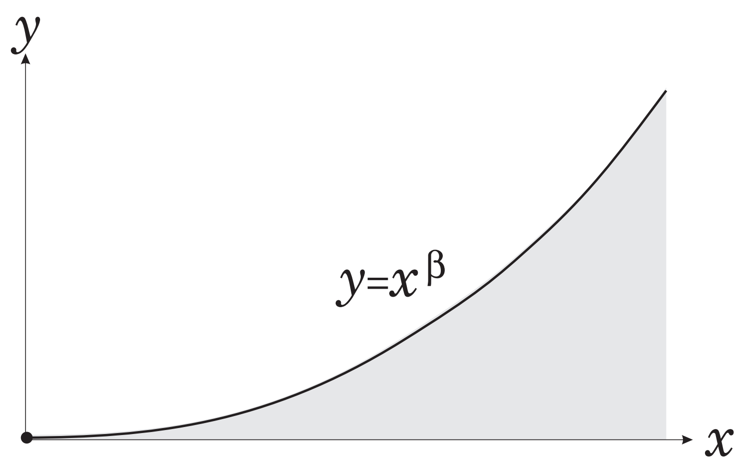
Finiteness theorems (Mostowski 85, Parusinski 94, Valette 05): There are **finitely many** outer Lipschitz equivalence classes in any definable family.

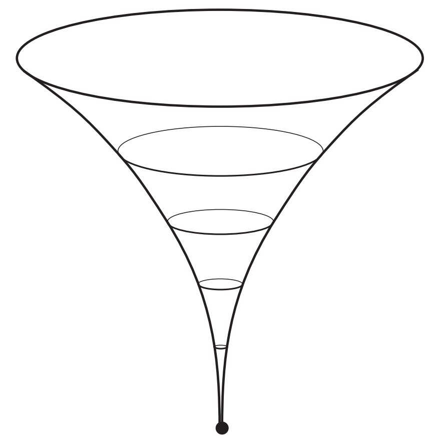
Inner Lipschitz classification of surface germs: Birbrair 99.

Our goal: outer Lipschitz classification of surface germs, i.e., canonical (unique up to outer Lipschitz equivalence) decomposition of a surface germ into normally embedded Hölder triangles.

Building blocks: For $\beta \in \mathbb{F}_{\geq 1}$, the **standard β-Hölder triangle** is the surface germ $T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}$.

The **standard β-horn** is $C_\beta = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$.





A **β-Hölder triangle** is a germ inner Lipschitz equivalent to T_β.

A **β-horn** is a germ inner Lipschitz equivalent to C_β.

Valette link

An **arc** γ in X is a map germ $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$.

The **Valette link** $V(X)$ is the space of all arcs in X.

The **tangency order** $tord(\gamma, \gamma')$ of arcs γ and γ' is the exponent $\kappa \in \mathbb{F}_{\geq 1} \cup \{\infty\}$, where $|\gamma(t) - \gamma'(t)| = ct^{\kappa} +$ (higher terms), $c \neq 0$. The tangency order defines a **non-archimedean metric** on $V(X)$.

A topologically non-singular arc $\gamma \in V(X)$ is **Lipschitz non-singular** if γ is an interior arc of a normally embedded Hölder triangle $T \subset X$. There are finitely many **Lipschitz singular** arcs in $V(X)$.

A Hölder triangle T is **non-singular** if all interior arcs of T are Lipschitz non-singular.

Zones in $V(X)$

A **zone** is a set $Z \subset V(X)$ such that for any arcs $\gamma \neq \gamma'$ in Z there is a non-singular Hölder triangle $T = T(\gamma, \gamma')$, bounded by γ and γ', such that $V(T) \subset Z$.

The **order** $\mu(Z)$ of a zone Z is the infimum of tangency orders of arcs in Z. A **singular** zone $Z = \{\gamma\}$ has order ∞.

A zone Z is **closed** if there are arcs γ and γ' in Z such that $itord(\gamma, \gamma') = \mu(Z)$, otherwise Z is **open**.

An arc γ in a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$, bounded by the arcs γ_1 and γ_2, is **generic** if $itord(\gamma, \gamma_1) = itord(\gamma, \gamma_2) = \beta$.

A zone Z is **perfect** if, for any $\gamma \neq \gamma'$ in Z, there is a Hölder triangle T such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T.

Pizza. Let a surface germ X be the union of a β-Hölder triangle T in the xy-plane and a graph $z = f(x, y)$ of a Lipschitz function f defined on T, such that $f(0, 0) = 0$.

The **order** $ord_\gamma f$ of f on $\gamma \subset T$ is $tord(\gamma, \gamma')$, where $\gamma' = (\gamma, f(\gamma))$. The set $Q(T)$ of exponents $q = ord_\gamma f$, for all $\gamma \subset T$, is either a closed interval in $\mathbb{F}_{\geq 1} \cup \{\infty\}$ or a point (a single exponent).

A β-Hölder triangle T is **elementary** with respect to f if each set $Z_q = \{\gamma \subset T,\ ord_\gamma f = q\}$, for $q \in Q(T)$, is a zone. If T is elementary, then $\mu(q) = \mu(Z_q)$ defines the **width function** $\mu : Q(T) \to \mathbb{F}_{\geq 1} \cup \{\infty\}$, such that $\beta \leq \mu(q) \leq q$.

T is a **pizza slice** for f if either $Q(T)$ is a point or the width function $\mu(q) = aq + b$ is a non-constant affine function on $Q(T)$.

If Q is not a point, then the boundary arc $\tilde{\gamma}$ of T where $\mu(q)$ is maximal is called the **supporting arc** of T.

A **pizza** on T associated with f is a decomposition of T into Hölder triangles T_j, each of them a pizza slice for f, with several **toppings:**

• exponent β_j of T_j,

• closed interval $Q_j = Q(T_j)$ in $\mathbb{F}_{\geq 1} \cup \{\infty\}$,

• affine width function $\mu_j(q) = a_j q + b_j$ on Q_j, where $\beta_j \leq \mu_j(q) \leq q$, or a single exponent $\mu_j = \beta_j$ when Q_j is a point,

• the supporting arc $\tilde{\gamma}_j$ of T_j when Q_j is not a point,

• the sign s_j of f on T_j (sign is not needed when f is non-negative).

A pizza is **minimal** if the union of any two adjacent pizza slices is not a pizza slice.

Theorem (Birbrair *et al.* 17). The minimal pizza exists and is unique, up to bi-Lipschitz equivalence, for the Lipschitz contact equivalence class of f.

For a non-negative Lipschitz function f on a normally embedded Hölder triangle T, the **Lipschitz contact equivalence class** of f is the same as the **outer Lipschitz equivalence class** of a surface germ $X = T \bigcup \{\text{graph of } f\}$.

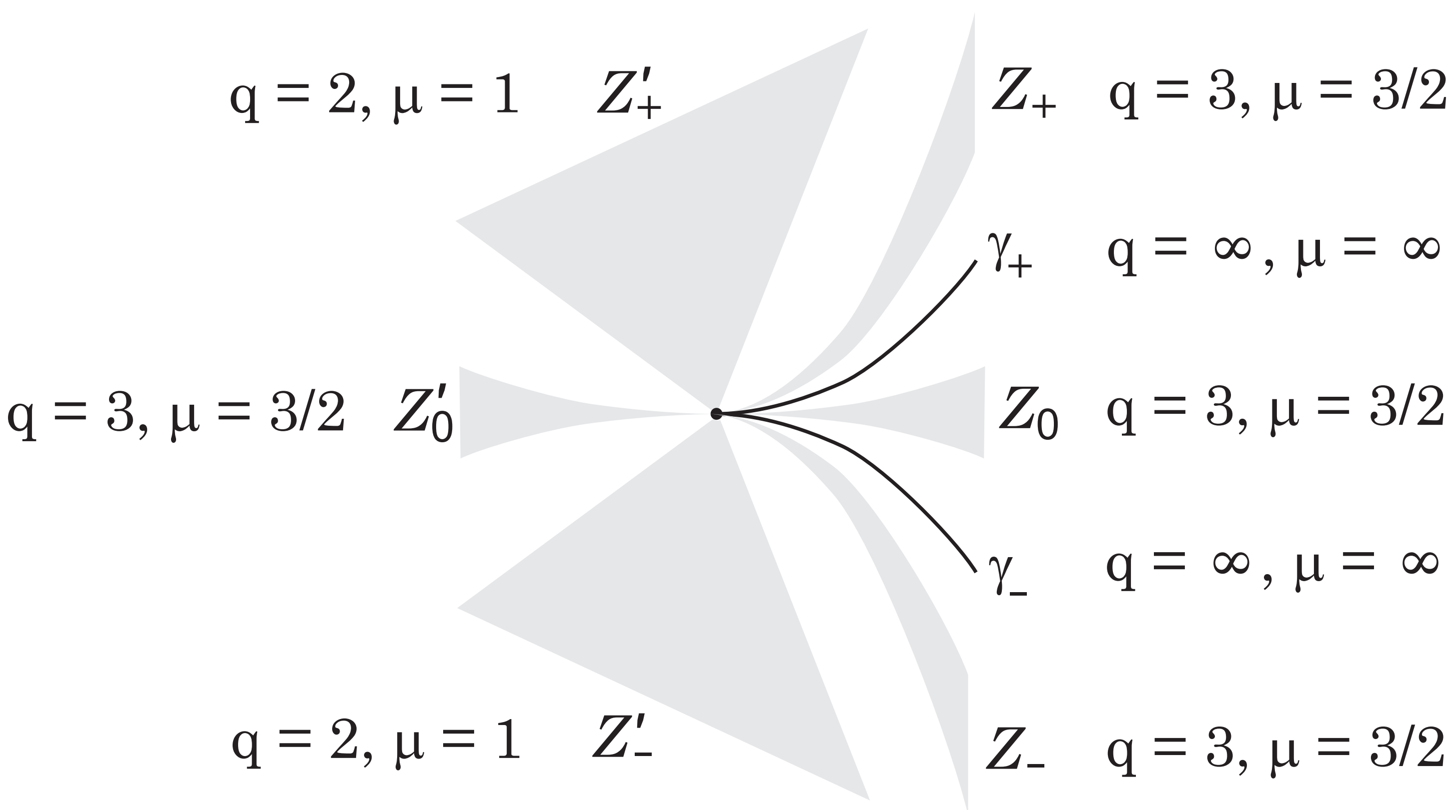
All toppings of a minimal pizza are **canonical**, while the pizza slices T_j are not. However, the boundary arcs of Hölder triangles T_j can be placed in **canonical** perfect zones $Z_i \subset V(T)$. **Here is the plan:**

1. Identify a **canonical** finite family of perfect **boundary zones** $Z_i \subset V(T)$ where boundary arcs of T_j can be placed.

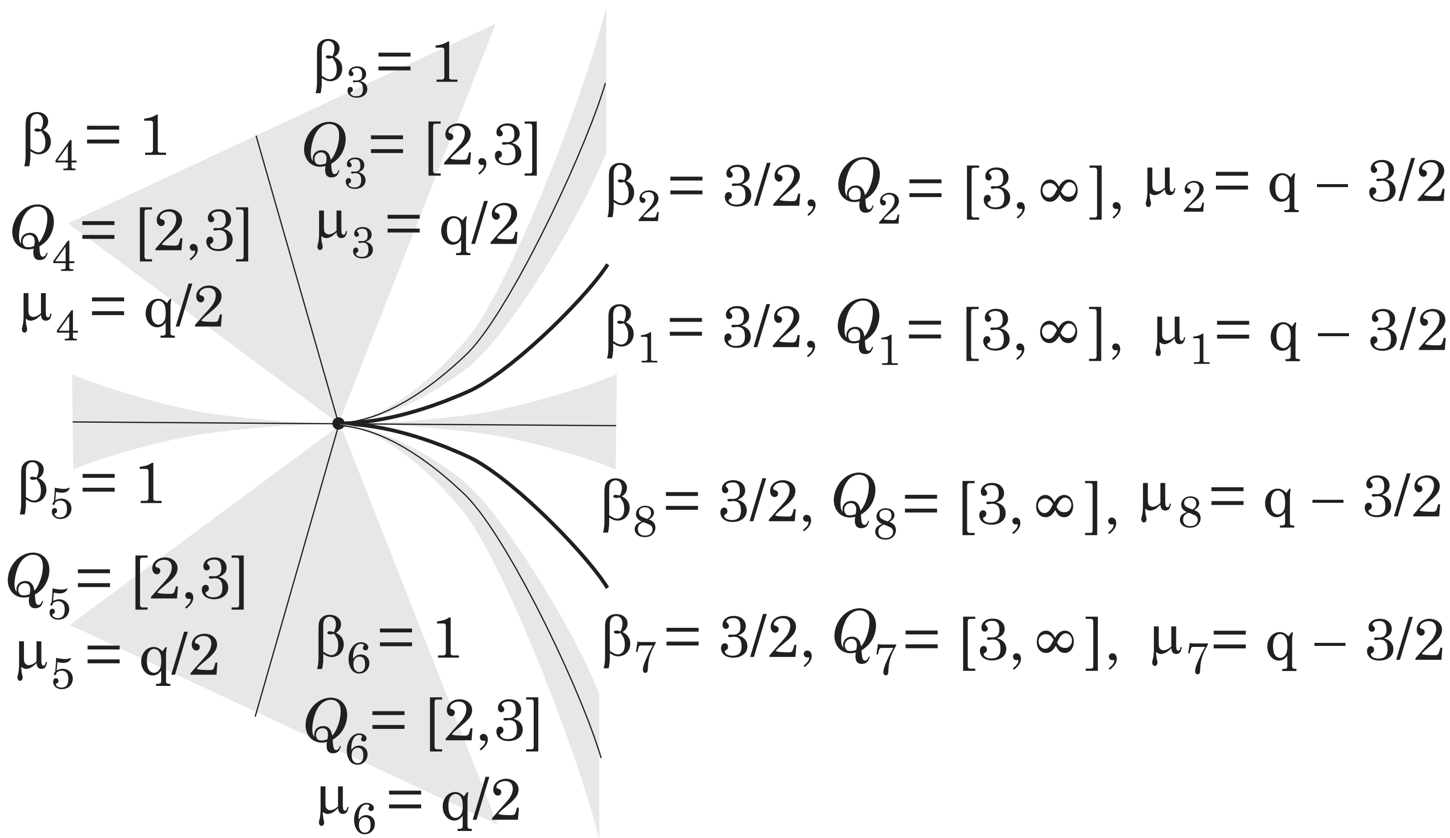
2. Choosing **arbitrary arcs** in the zones Z_i, define a decomposition of T into Hölder triangles T_j. All choices define minimal pizzas for f, resulting in **outer Lipschitz equivalent** decompositions of X.

3. Replacing this geometric construction with an **abstract pizza**, a combinatorial object, we get an **outer Lipschitz invariant** of X.

Example: $f(x,y) = y^2 - x^3$. We have $f|_{\gamma_\pm} \equiv 0$, where $\gamma_\pm = \{x \geq 0,\ y = \pm x^{3/2}\}$ are **singular boundary zones**. There are six **boundary zones** of finite order μ:



A minimal pizza for f consists of eight slices T_j with the boundary arcs γ_+, γ_- and an arbitrary arc selected in each of the six other boundary zones.



General surface germ X: normal and abnormal zones.

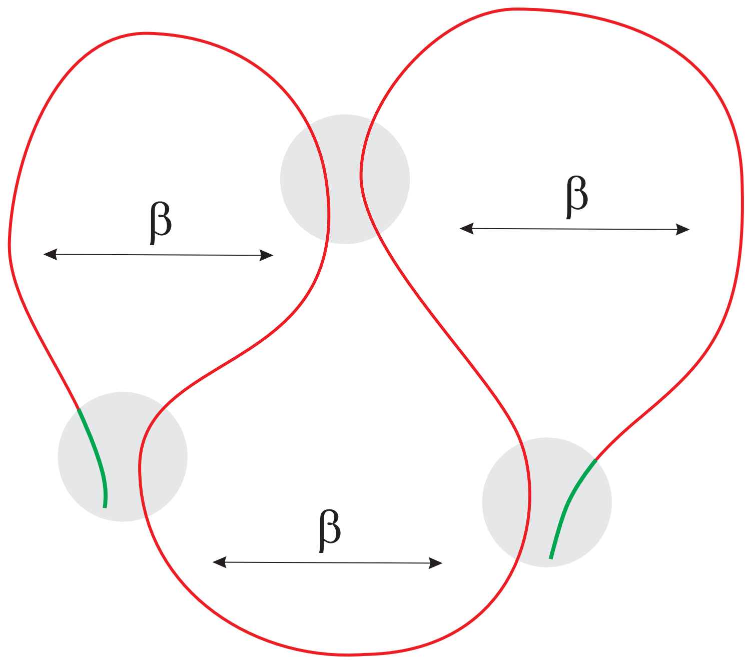
A Lipschitz non-singular arc $\gamma \subset X$ is **abnormal** if there are normally embedded Hölder triangles T and T' in X such that $\gamma = T \cap T'$ and $T \cup T'$ is not normally embedded. Otherwise, γ is **normal**.

A zone $Z \subset V(X)$ is **abnormal** (resp., **normal**) if all arcs in Z are abnormal (resp., normal). An abnormal (resp., normal) zone is **maximal** if it is not contained in a larger abnormal (resp., normal) zone.

Theorem (AG, Souza 21) For any surface germ X, there is a **canonical partition** of $V(X)$ into **maximal abnormal** zones and **maximal normal** zones.
All maximal normal zones are normally embedded.
All maximal abnormal zones are perfect. Each of them is either a normally embedded **non-snake zone**, or a **snake zone**: a disjoint union of normally embedded **segment zones** and **nodal zones**.

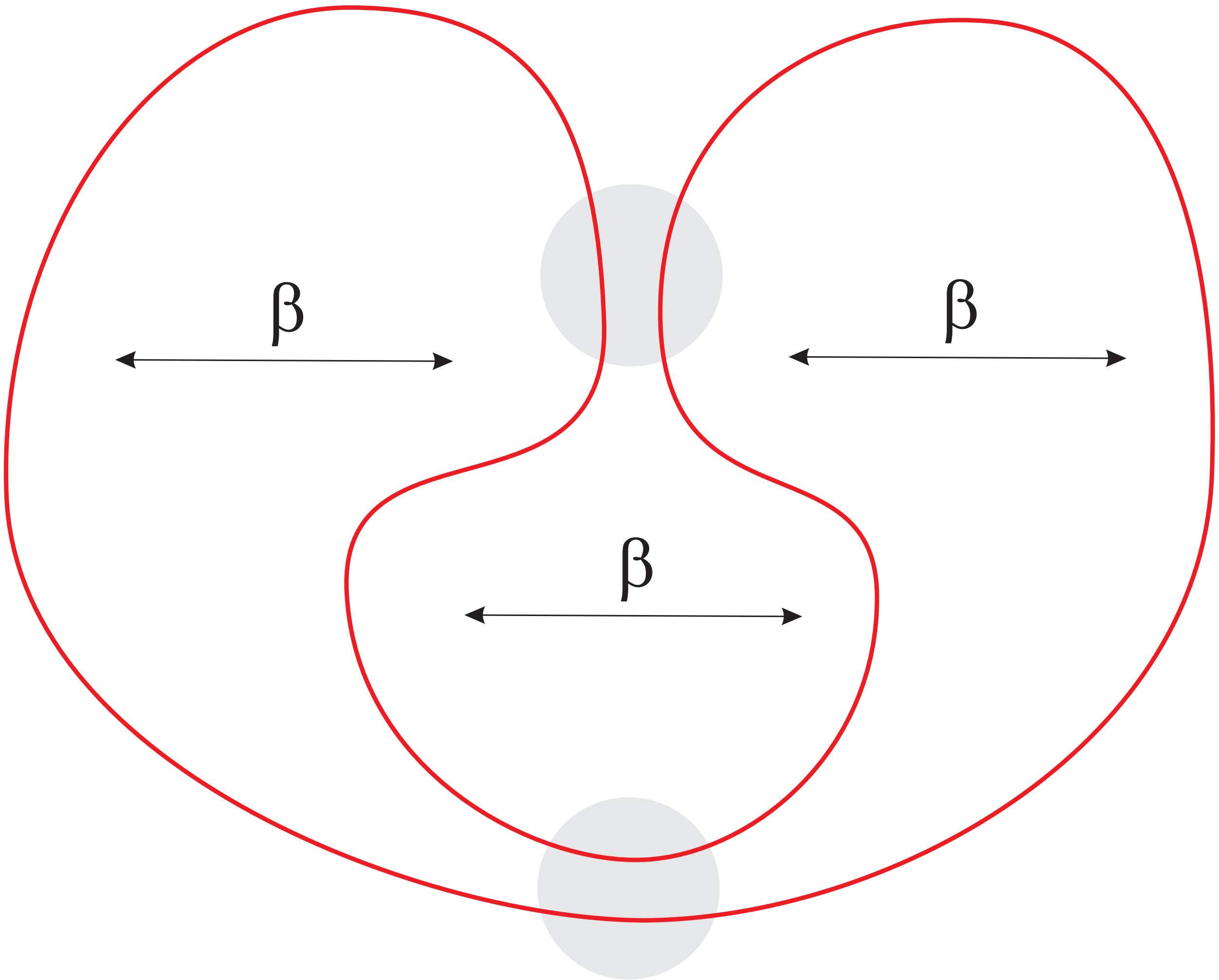
Snakes, circular snakes, bubble snakes and non-snake bubbles.

A β-**snake** is a non-singular β-Hölder triangle T such that each generic arc in $V(T)$ is abnormal.
A maximal abnormal β-zone $Z \subset V(X)$ is a **snake zone** if there is a β-snake $T \subset X$ such that Z is the set of generic arcs of T.



The link of a β-snake. Shaded disks indicate nodal zones.

A **circular β-snake** is a β-horn C such that all arcs in $V(C)$ are abnormal. The set $V(C)$ is called a **circular β-snake zone**.

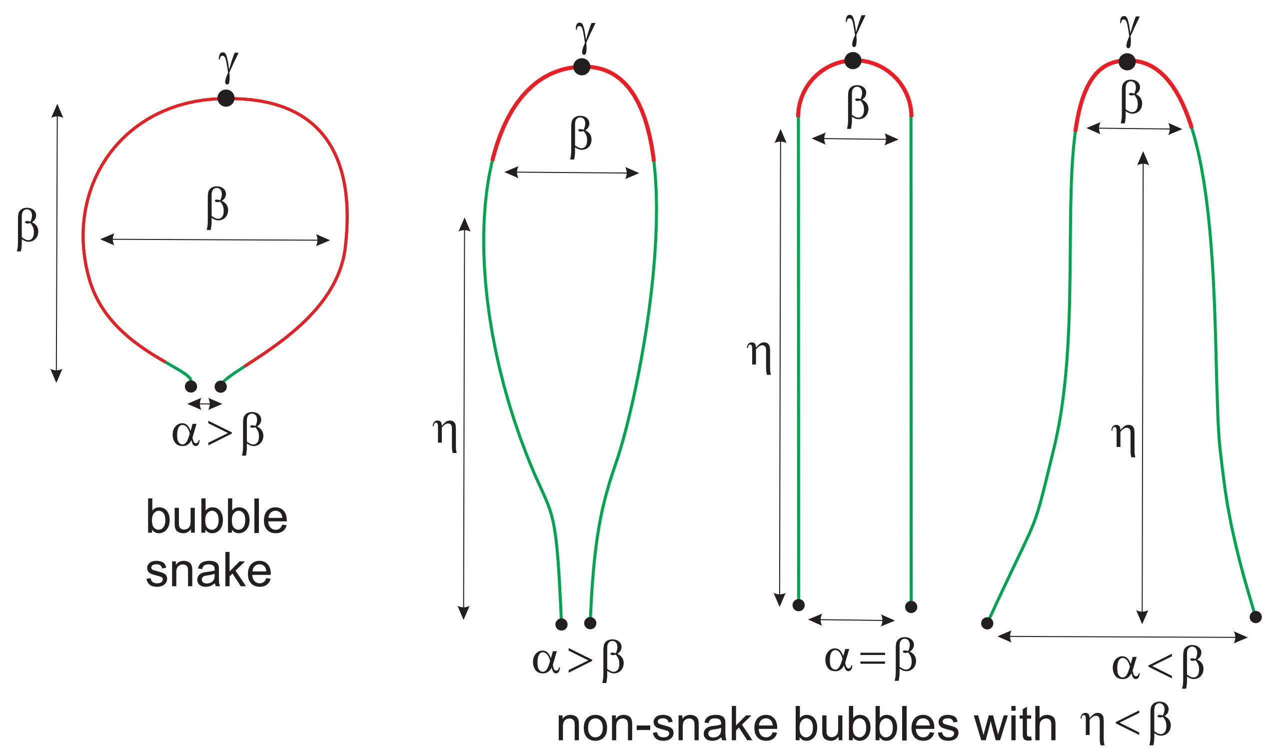


The link of a circular β-snake. Shaded disks indicate nodal zones.

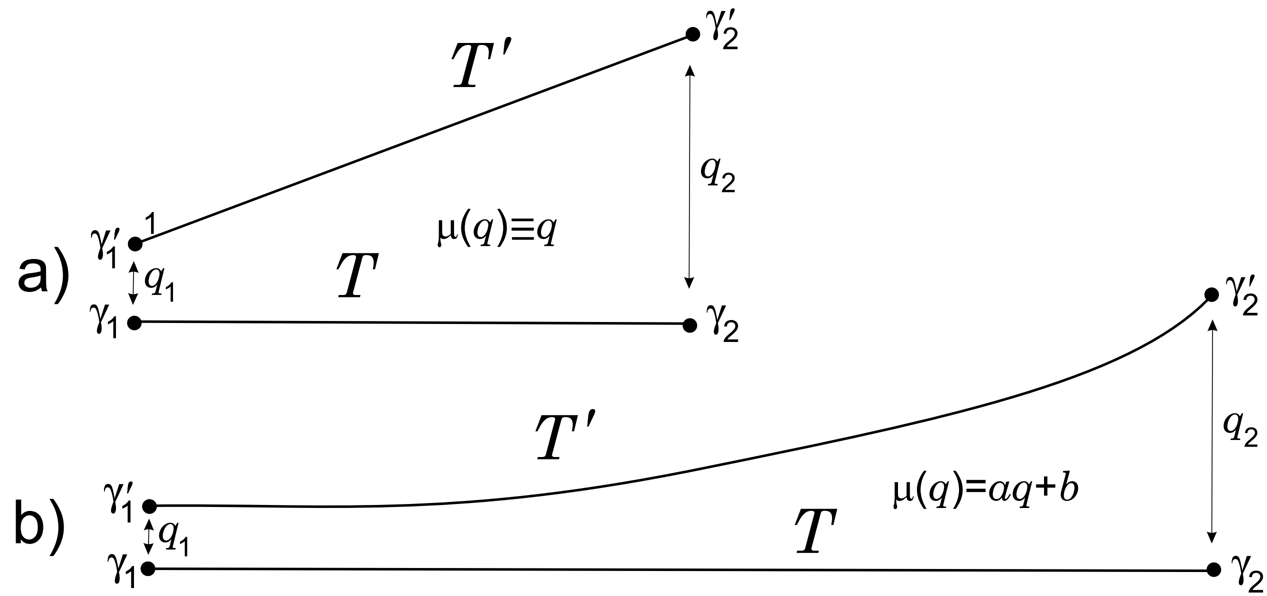
A **bubble** is a Hölder triangle T bounded by γ_1 and γ_2, such that $tord(\gamma_1, \gamma_2) > itord(\gamma_1, \gamma_2)$, partitioned into normally embedded triangles by an arc γ. A **bubble snake** is a bubble that is a snake.

A **non-snake bubble** is a bubble that does not contain a snake.

A **non-snake abnormal zone** is a maximal abnormal zone $Z \subset V(T)$ where T is a non-snake bubble.



Transverse and coherent pairs of Hölder triangles



A pair (T, T') of normally embedded Hölder triangles is **transverse** if $T \cup T'$ is a subset of a normally embedded Hölder triangle.

A pair (T, T') of normally embedded Hölder triangles is **coherent** if it is outer Lipschitz equivalent to the union of T and a graph T' of a Lipschitz function f defined on T, such that T is a pizza slice for f with $\mu(q) = aq + b \not\equiv q$.

Outer Lipschitz invariant decomposition of the Valette link of a surface germ X.

Step 1. Define canonical **fundamental zones** in $V(X)$:

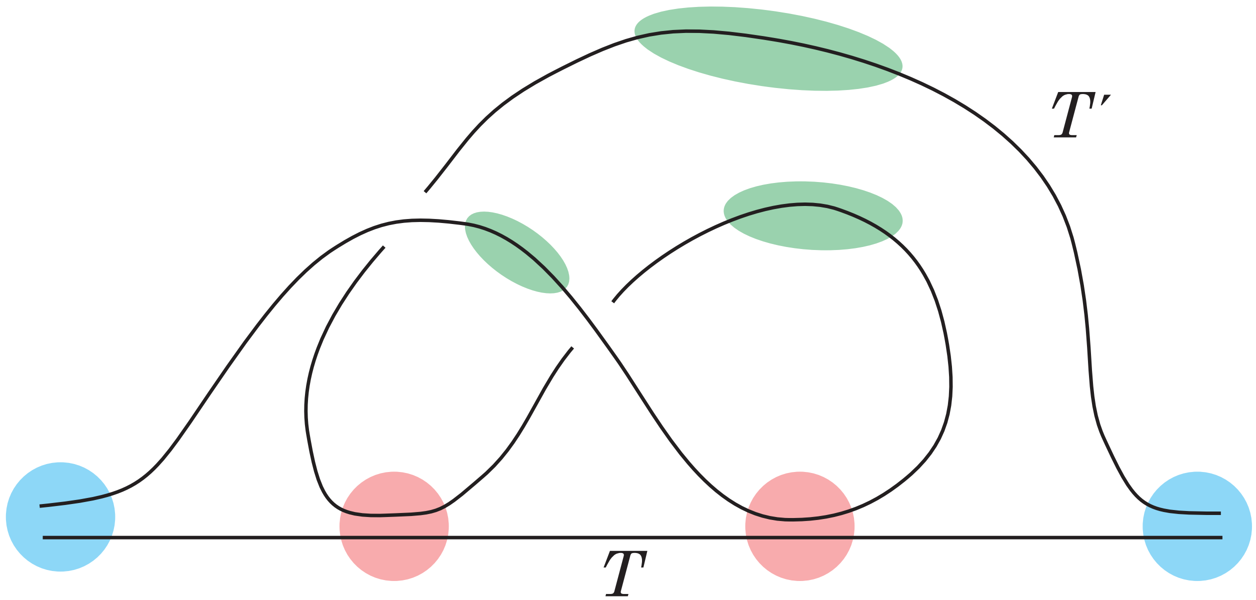
- Lipschitz singular arcs,
- Maximal normal zones,
- Segment and nodal zones of snakes and circular snakes,
- Non-snake abnormal zones.

Step 2. Using pizza decompositions for the "distance functions" between fundamental zones, define **global pizza zones** in $V(X)$ as minimal by inclusion zones among pizza zones of these pizza decompositions, their non-empty intersections, and their intersections with abnormal fundamental zones. All global pizza zones are perfect.

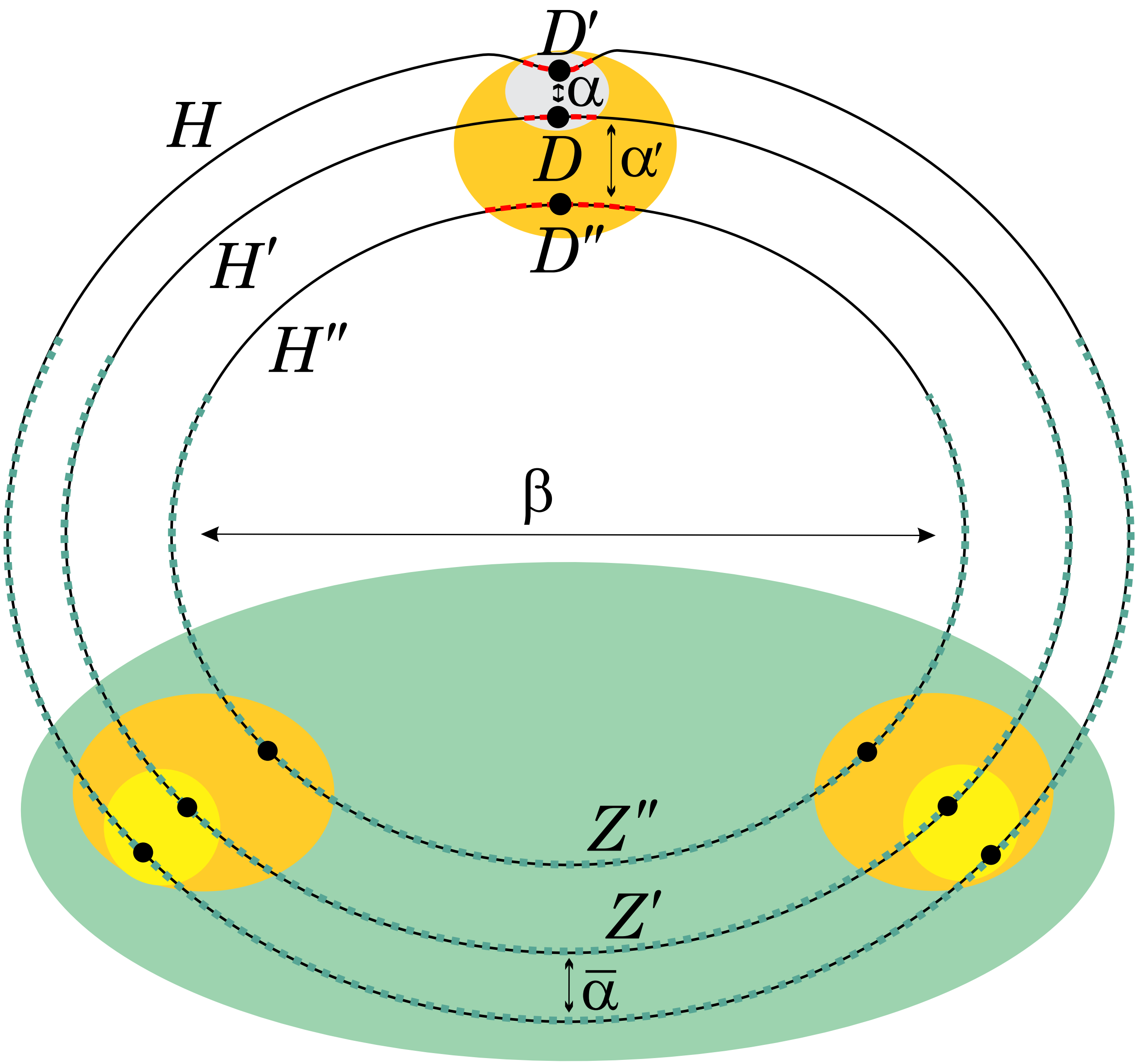
Step 3. Placing **boundary arcs** in global pizza zones, decompose X into finitely many isolated arcs and normally embedded Hölder triangles, so that any two Hölder triangles are either coherent or transverse, and all choices of arcs result in outer Lipschitz equivalent decompositions.
It may be necessary to place two or three boundary arcs in some of the global pizza zones.

Step 4. Define **combinatorial equivalence** of these decompositions, so that two surface germs are **outer Lipschitz equivalent** iff the corresponding decompositions are **combinatorially equivalent**.

This program has been completed for the union of two normally embedded Hölder triangles (Birbrair, AG 25).



Example: Pizza zones for the union of two normally embedded Hölder triangles.



Example: Decomposition of a three-horn surface germ

Normal pairs of normally embedded Hölder Triangles

Given two Hölder triangles T and T', a pair of arcs $\gamma \subset T$ and $\gamma' \subset T'$, is **normal** if $tord(\gamma, T') = tord(\gamma, \gamma') = tord(\gamma', T)$.
A pair (T, T') of normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ is **normal** if both pairs (γ_1, γ'_1) and (γ_2, γ'_2) of their boundary arcs are normal.

Example. If Γ is a graph of a Lipschitz function f on T, then any pair of arcs (γ, γ'), where $\gamma \subset T$ and $\gamma' \subset \Gamma$ is the graph of $f|_\gamma$, is normal, and the pair (T, Γ) of Hölder triangles is normal.

Theorem (Birbrair, AG 22). Let (T, T') be a normal pair of normally embedded Hölder triangles, such that T is elementary with respect to $f(x) = dist(x, T')$. Then (T, T') is outer Lipschitz equivalent to a pair (T, Γ), where Γ is the graph of f. Moreover, T' is elementary with respect to $g(x') = dist(x', T)$, and a minimal pizza for g on T' is equivalent to a minimal pizza for f on T.

Maximum zones

Let (T, T') be a normal pair of normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$. Let $\{D_\ell\}_{\ell=0}^p$ be the pizza zones in $V(T)$ of a minimal pizza on T associated with $f(x) = dist(x, T')$, ordered from $D_0 = \{\gamma_1\}$ to $D_p = \{\gamma_2\}$. The **exponent** $q_\ell = tord(D_\ell, T')$ of the zone D_ℓ is defined as $ord_\gamma f$ for $\gamma \in D_\ell$ (it is the same for all $\gamma \in D_\ell$).

A zone D_ℓ is a **maximum zone** if either $0 < \ell < p$ and $q_\ell \geq \max(q_{\ell-1}, q_{\ell+1})$, or $\ell = 0$ and $q_0 \geq q_1$, or $\ell = p$ and $q_p \geq q_{p-1}$.

Maximum zones in $V(T')$ are some of the pizza zones D'_ℓ of a minimal pizza on T' associated with $g(x') = dist(x', T)$. They are defined similarly, exchanging T and T'.

Theorem (Birbrair, AG 22). Let (T, T') be a normal pair of normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ oriented from γ_1 to γ_2 and from γ'_1 to γ'_2, respectively.

Let $\{M_i\}_{i=1}^m$ and $\{M'_j\}_{j=1}^n$ be the maximum zones in $V(T)$ and $V(T')$, ordered according to the orientations of T and T'.
Let $\bar{q}_i = tord(M_i, T')$ and $\bar{q}'_j = tord(M'_j, T)$.

Then $m = n$, and there is a canonical permutation

$$\sigma : [1, \ldots, m] \to [1, \ldots, m]$$

such that $ord(M_i) = ord(M'_{\sigma(i)})$ and $tord(M_i, M'_{\sigma(i)}) = \bar{q}_i = \bar{q}'_{\sigma(i)}$.

If $\{\gamma_1\} = M_1$ is a maximum zone, then $\{\gamma'_1\} = M'_1$ and $\sigma(1) = 1$. If $\{\gamma_2\} = M_m$ is a maximum zone, then $\{\gamma'_2\} = M'_m$ and $\sigma(m) = m$.

Transverse and coherent pairs of Hölder triangles

A pair (T, T') of normally embedded Hölder triangles is **transverse** if $T \cup T'$ is a subset of a normally embedded Hölder triangle.

A non-transverse pair (T, T') of normally embedded Hölder triangles is **coherent** if it is outer Lipschitz equivalent to the union of T and a graph T' of a Lipschitz function f defined on T, such that T is a pizza slice for f.

A pizza slice T_j of a pizza decomposition of a Hölder triangle T associated with a function f is **transverse** if T_j and the graph of $f|_{T_j}$ are transverse Hölder triangles.

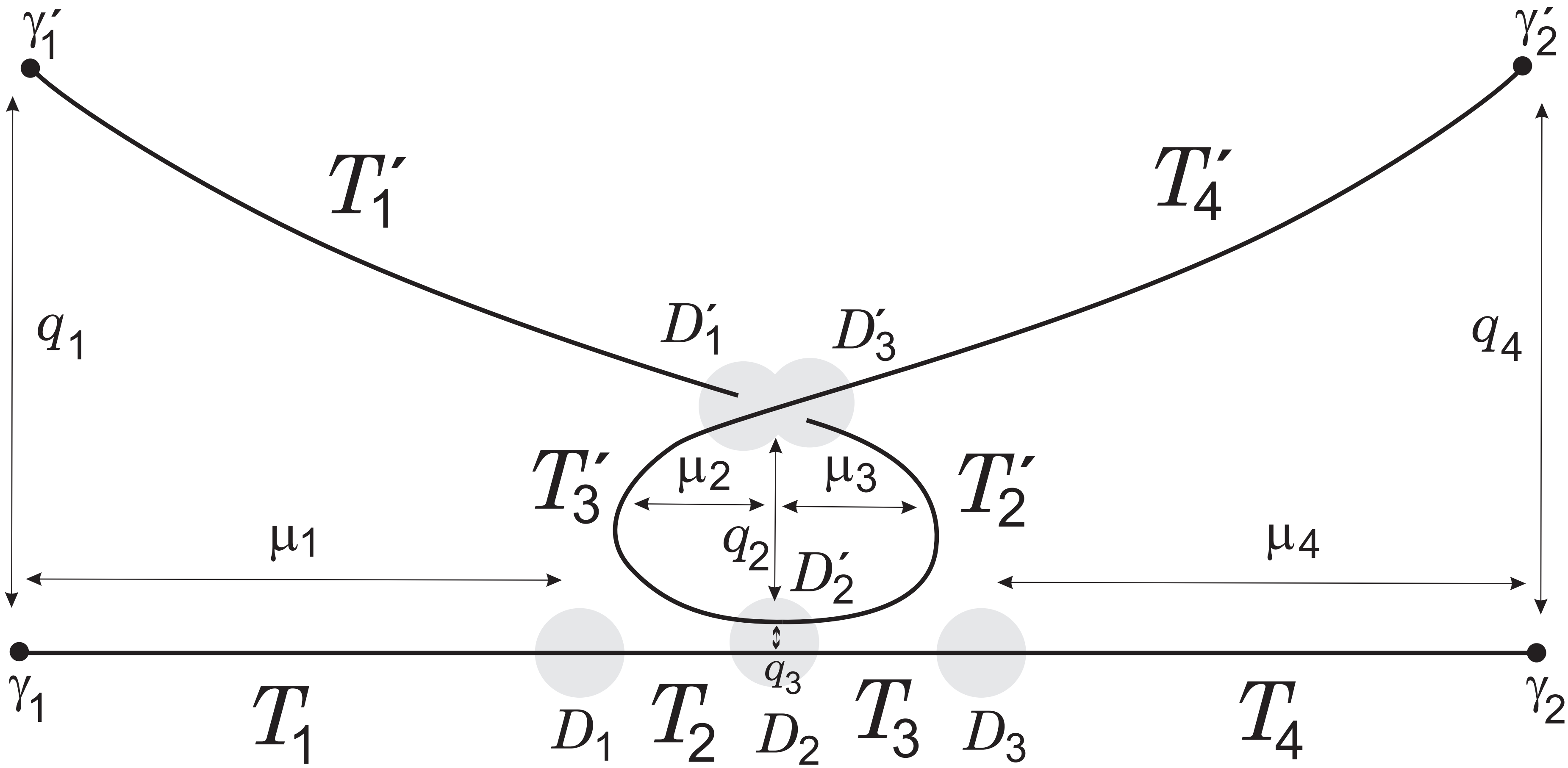
Alternatively, a pizza slice T_j is transverse if $\mu_j(q) \equiv q$, where μ_j is the affine width function on Q_j associated with f.

A pizza slice T_j is **coherent** if it is not transverse, thus $\mu_j(q) \not\equiv q$.

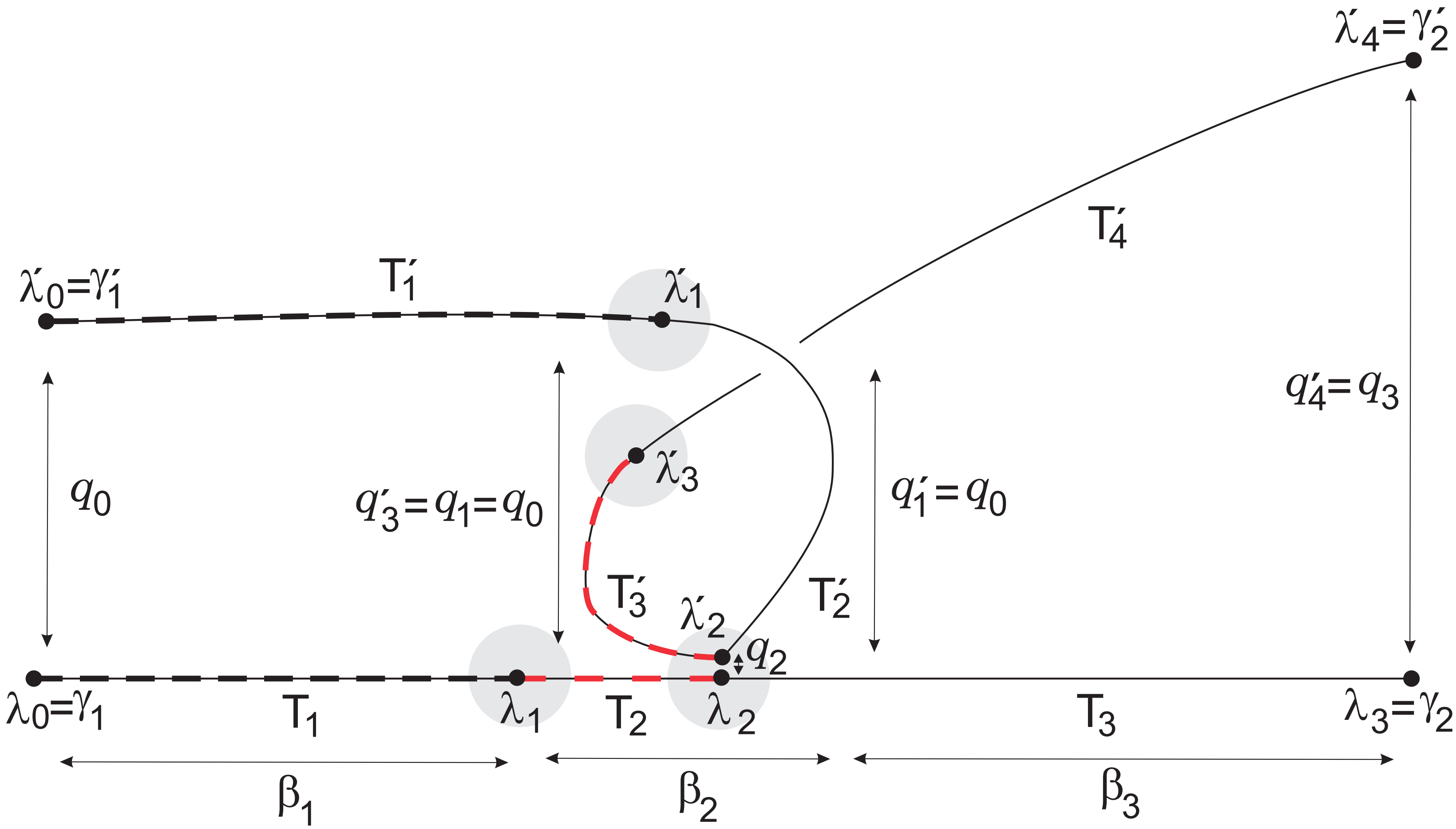
Theorem (Birbrair, AG 22). Let (T, T') be a normal pair of normally embedded Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2, respectively. Let $\{T_i\}_{i=1}^p$ and $\{T'_j\}_{j=1}^s$ be minimal pizza decompositions of T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$.

Then there is a canonical one-to-one correspondence $j = \tau(i)$ between coherent pizza slices T_i of T and coherent pizza slices T'_j of T', such that each pair (T_i, T'_j), where $j = \tau(i)$, is outer Lipschitz equivalent to the pairs (T_i, Γ) and (T'_j, Γ'), where Γ is the graph of $f|_{T_i}$ and Γ' is the graph of $g|_{T'_j}$.

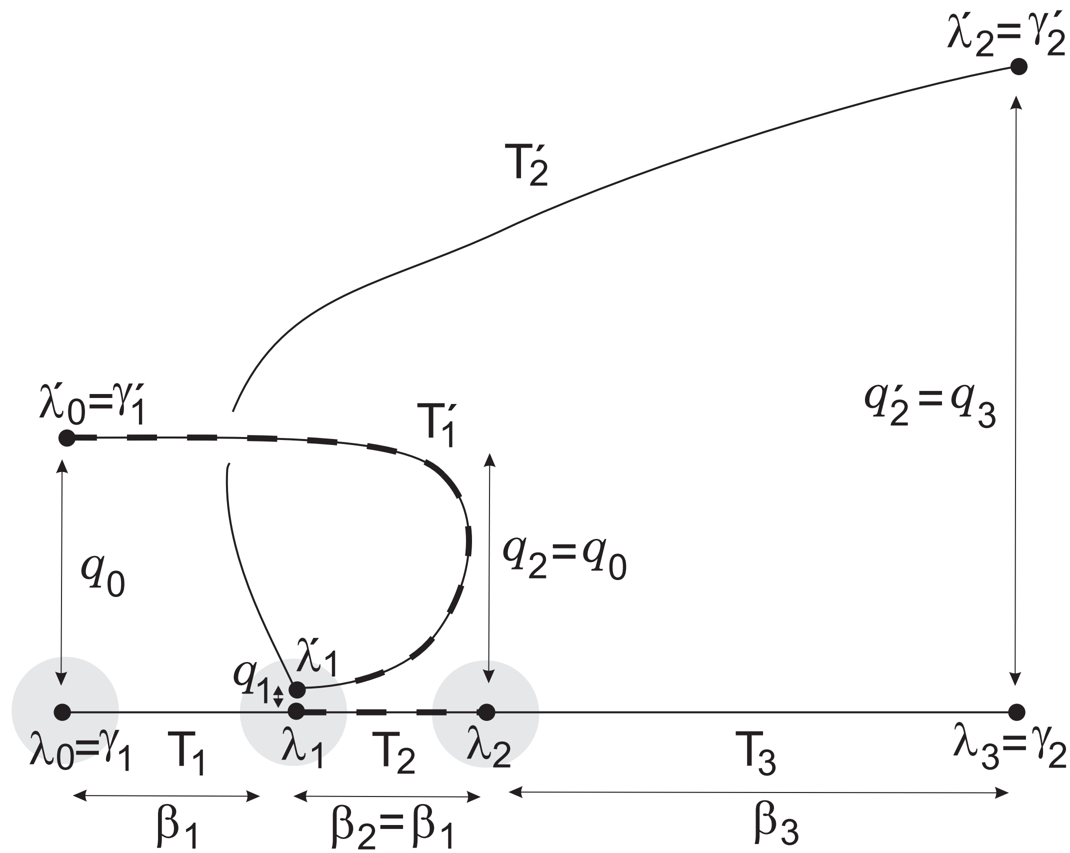
The pair (T_i, T'_j), where $j = \tau(i)$, is **positive** if orientations of T_i and T'_j induced by the correspondence τ are consistent with orientations of T and T'. Otherwise, the pair (T_i, T'_j) is **negative**.
Thus τ is a **signed** correspondence.



Example: A normal pair of normally embedded Hölder triangles with two positive pairs (T_1, T_1') and (T_4, T_4') of coherent pizza slices, and two negative pairs (T_2, T_3') and (T_3, T_2').



Example: A normal pair of Hölder triangles with different number of pizza slices in the minimal pizzas on T and T' associated with the distance functions $f = dist(x, T')$ and $g = dist(x', T)$.



Example: A normal pair of Hölder triangles with different number of pizza slices in the minimal pizzas on T and T' associated with the distance functions $f = dist(x, T')$ and $g = dist(x', T)$.

Theorem (Birbrair, AG 25). Two normal pairs (T,T') and (S,S') of normally embedded Hölder triangles are outer Lipschitz equivalent if, and only if, the following holds:

1. The minimal pizzas on T and T' associated with the distance functions $f(x) = dist(x,T')$ and $g(x') = dist(x',T)$ are equivalent to the minimal pizzas on S and S' associated with the distance functions $\phi(s) = dist(s,S')$ and $\psi(s') = dist(s',S)$, respectively.

2. The numbers of maximum zones for the pairs (T,T') and (S,S') are equal, and the permutation σ of the maximum zones for the pair (S,S') is the same as for the pair (T,T').

3. The numbers of coherent pizza slices for the pairs (T,T') and (S,S') are equal, and the signed correspondence τ between coherent pizza slices for the pair (S,S') is the same as for the pair (T,T').