Schwartz-MacPherson classes in the Lipschitz framework

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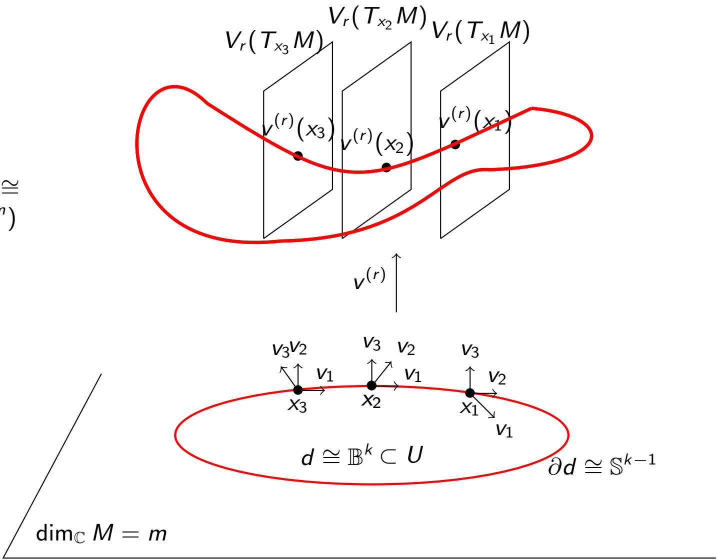
Jagiellonian University - Kraków.

$V_r(T_xM)$ is the set of all complex r-frames tangent to M at x.
The complex Stiefel manifold $V_r(\mathbb{C}^m)$ is the set of all complex r-frames in $\mathbb{C}^m$.

$TM|_U \cong U \times \mathbb{C}^m$

$V_r(TM)|_U \cong U \times V_r(\mathbb{C}^m)$



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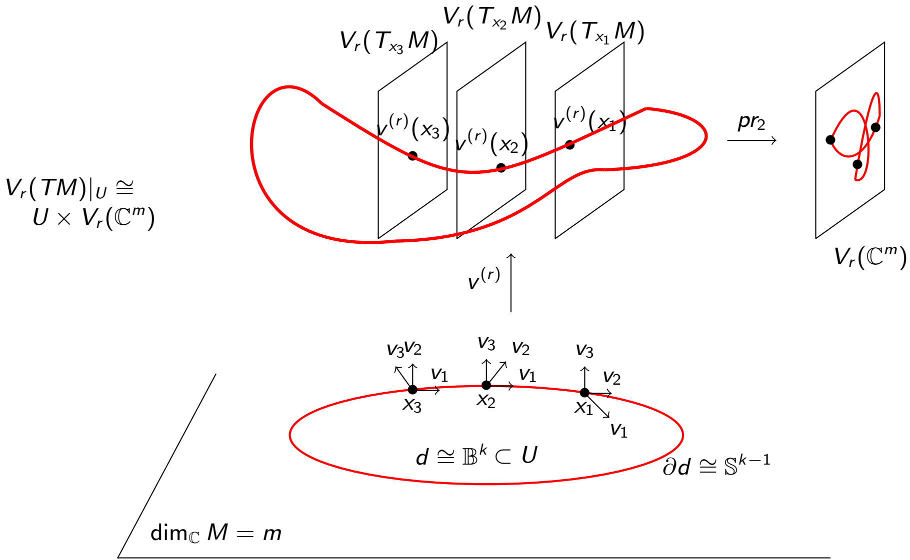


An r-frame $v^{(r)}$ defined on the boundary ∂d of a k-cell d provides a map:

$$\partial d \cong \mathbb{S}^{k-1} \xrightarrow{v^{(r)}} V_r(TM)|_U \cong U \times V_r(\mathbb{C}^m) \xrightarrow{pr_2} V_r(\mathbb{C}^m).$$

The map

$$\partial d \cong \mathbb{S}^{k-1} \overset{pr_2 \circ v^{(r)}}{\longrightarrow} V_r(\mathbb{C}^m)$$

defines an element $[\xi(v^{(r)}, d)]$ of $\pi_{k-1}(V_r(\mathbb{C}^m))$.

Suppose that $[\xi(v^{(r)}, d)] = 0$, then, by classical homotopy theory, the map $\mathbb{S}^{k-1} \to V_r(\mathbb{C}^m)$ defined on $\mathbb{S}^{k-1}$ can be extended inside the ball $\mathbb{B}^k$.
In that case, that means that the r-frame, $v^{(r)}$ can be extended inside the cell d.
There is no obstruction to the extension of the section $v^{(r)}$ inside d.

$$\begin{array}{ccc} \partial d \cong \mathbb{S}^{k-1} & \longrightarrow & V_r(\mathbb{C}^m) \\ \downarrow & \nearrow & \\ d \cong \mathbb{B}^k & & \end{array}$$

This happens for example in the case $\pi_{k-1}(V_r(\mathbb{C}^m)) = 0$.

Index of an r-frame

One has:

$$\pi_{k-1}(V_r(\mathbb{C}^m)) = \begin{cases} 0 & \text{for } k < 2m-2r+2 \\ \mathbb{Z} & \text{for } k = 2m-2r+2 \end{cases}$$

One can construct an r-frame by choosing any r-frame $v^{(r)}$ on the 0-skeleton of the cell decomposition (D), then one can extend it without singularities till the obstruction dimension

$$2p = 2(m-r+1).$$

The r-frame $v^{(r)}$ has no singularity on the $(2p-1)$-skeleton and isolated singularities on the $2p$-skeleton of (D). Given the r-frame $v^{(r)}$ on the boundary of each $2p$-cell d, one can extend $v^{(r)}$ on d with a singularity at the barycenter $\hat{d}$ of index $I(v^{(r)}, \hat{d})$ defined by

$$I(v^{(r)}, \hat{d}) = [\xi(v^{(r)}, d)] \in \pi_{2p-1}(V_r(\mathbb{C}^m)) = \mathbb{Z}.$$

The Chern classes.

• The r-frame is constructed on the boundaries of $2p$-cells and can be extended inside them, for instance by homothety, with a singular point at the barycenter of these cells.
• That defines an index for each $2p$-dimensional cell.

The $2p$-cochain

$$\gamma : d \mapsto I(v^{(r)}, \hat{d}) \qquad \gamma = \sum_{d \in (D)^{2p}} I(v^{(r)}, \hat{d})\, d^* \in C^{2p}(D, \mathbb{Z}),$$

where d^* is the elementary (D)-cochain whose value is 1 at the cell d and 0 at all other cells, is actually a cocycle. Its cohomology class is the p-th Chern class of M

$$c^p(TM) \in H^{2p}(M; \mathbb{Z}).$$

The Schwartz classes.

The first idea of Marie-Hélène Schwartz is to consider "radial" extensions of vector fields and frames. She performed these extensions in the framework of Whitney stratifications, we will see that in the framework of Lipschitz stratifications the constructions is simpler.

The second idea of Marie-Hélène Schwartz concerns the obstruction dimension. Using a cellular decomposition (in the ambient manifold) dual of a triangulation compatible with the given stratification allows to define a suitable global cocycle.

Stratifications

Let X be an analytic set, closed subset of an analytic manifold M. we consider a *filtration*:

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X = X_n$$

such that each *stratum*, difference $V_\alpha = X_\alpha - X_{\alpha-1}$, is either empty or analytic manifold of pure (complex) dimension α.
The stratifications we consider satisfy the *frontier condition*:

$$V_\alpha \cap \overline{V}_\beta \neq \emptyset \Rightarrow V_\alpha \subset \overline{V}_\beta$$

The obstruction dimensions.

The obstruction dimension to the construction of an r-frame tangent to M is equal to $2p = 2(m - r + 1)$.
The obstruction dimension to the construction of an r-frame tangent to a stratum V_α of complex dimension s_α is equal to $2p_\alpha = 2(s_\alpha - r + 1)$.

Using a triangulation (K) compatible with the stratification does not allow to define an obstruction cocycle of same dimension for all strata.

The use of dual cells.

The observation of M.-H. Schwartz concerns the obstruction dimensions.
Let (K) be a triangulation of M compatible with the stratification of (M, X). The cells of a cellular decomposition, dual to (K), have the particular property of being transverse to the strata.
The property implies that:

The dimension of the intersection of a $2p$-dimensional cell d^{2p} with a stratum V_α of dimension $2s_\alpha$ is precisely the obstruction dimension for the construction of an r-frame tangent to V_α. That is $2p_\alpha = 2(s_\alpha - r + 1)$.

	complex dimension	intersection with d^{2p}	obstruction dimension
M	m	$2p$	$2p = 2(m - r + 1)$
X_{reg}	n	$2p_n = 2p - 2(m - n)$	$2p_n = 2(n - r + 1)$
V_α	s_α	$2p_\alpha = 2p - 2(m - s_\alpha)$	$2p_\alpha = 2(s_\alpha - r + 1)$

Lipschitz stratifications.

T. Mostowski (complex analytic varieties)
A. Parusiński (real analytic varieties)

- c is a constant $c > 1$,
- $q = q_{\alpha_1}$ is a point in a α_1-dimensional stratum V_{α_1}

A chain is a sequence of points $q_{\alpha_s} \in V_{\alpha_s}$, such that

$$X_{\alpha_1} \supset X_{\alpha_2} \supset \cdots \supset X_{\alpha_s} \supset \cdots X_{\alpha_r} = X_\ell$$

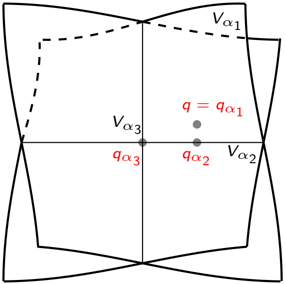
(with dimensions $\alpha_1 > \alpha_2 > \cdots > \alpha_s > \cdots > \alpha_r = \ell$)
and α_s is the bigger integer such that

$d(q, X_{\alpha_k}) \geq 2c^2\, d(q, X_{\alpha_s})$ for all k for which $\alpha_s > \alpha_k \geq \ell$

and

$|q - q_{\alpha_s}| \leq c\, d(q, X_{\alpha_s})$.

The notion of chain. (Mostowski).

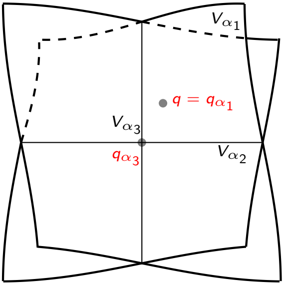


Here, $\alpha_1 = 2, \alpha_2 = \alpha_s = 1, \alpha_3 = 0,$

$$d(q, X_{\alpha_3}) \geq 2c^2\, d(q, X_{\alpha_2})$$

$$|q - q_{\alpha_2}| \leq c\; d(q, X_{\alpha_2}).$$

Chain: $q_{\alpha_1},\; q_{\alpha_2},\; q_{\alpha_3}$



Here, $\alpha_1 = 2, \alpha_3 = \alpha_s = 0,$

$$|q - q_{\alpha_3}| \leq c\; d(q, X_{\alpha_3}).$$

Chain: $q_{\alpha_1},\; q_{\alpha_3}$

Lipschitz stratifications.

For $q \in V_j$,
$P_q : \mathbb{R}^n \to T_q V_j$ be the orthogonal projection onto the tangent space
$P_q^\perp = Id - P_q$ the orthogonal projection onto the normal space $T_q^\perp V_j$.

The stratification is *L-stratification* if, for some constant $C > 0$ and every chain $q = q_{\alpha_1}, q_{\alpha_2}, \dots, q_{\alpha_r}$ and every k, $1 \leq k \leq r$,

$$|P^\perp_{q_{\alpha_1}} P_{q_{\alpha_2}} \cdots P_{q_{\alpha_k}}| \leq C|q_{\alpha_1}, -q_{\alpha_2}|/d(q_{\alpha_1}, X_{\alpha_k - 1})$$

If, further, $q' \in V_{\alpha_1}$ and $|q' - q| \leq (1/2c)\, d(q, X_{\alpha_1 - 1})$, then

$$|(P_{q'} - P_q)P_{q_{\alpha_2}} \cdots P_{q_{\alpha_k}}| \leq C|q' - q|\ /d(q, X_{\alpha_k - 1}),$$

in particular (for q and q' in V_{α_1}),

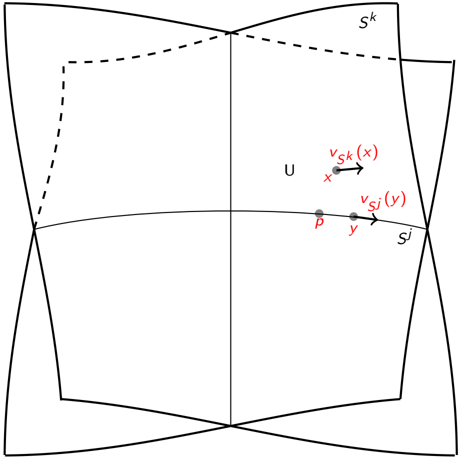
$$|P_{q'} - P_q| \leq C|q' - q|\ /d(q, X_{\alpha_1 - 1}).$$

Let X stratified subset of the smooth manifold M.

A stratified vector field $v = \{v_j = v|_{V_j}\}$ is said rugose
if for each point $p \in V_j$, there is a constant $C > 0$ and a neighbourhood U of p in M such that for each point $y \in V_j \cap U$, and each point $x \in X \cap U$, if V_k denotes the stratum of X containing x, then

$$\|v_k(x) - v_j(y)\| < C|x - y|.$$

For v to be Lipschitz
one need to allow y not only
to belong to V_j
but also to any stratum
incident to V_j.



Lipschitz (L) $\Rightarrow$ Kuo-Verdier (w) $\Rightarrow$ Whitney (b) $\Rightarrow$ Whitney (a).

Extension of Lipschitz frames

Proposition (Mostowski (1985))

Let $\{V_j\}_{j=1}^{m}$ be a Lipschitz-stratification of X and let v be a Lipschitz stratified vector field on X_j, $(\ell \leq j \leq m)$. Then v can be extended to a Lipschitz stratified vector field on V_{j+1} (parallel extension).

Corollary

If v_1 and v_2 are $\mathbb{C}$-linearly independent on a stratum V_α, then their Lipschitz parallel extensions are $\mathbb{C}$-linearly independent in a neighbourhood of V_α.

In a more general way, one has:

Corollary (Result A)

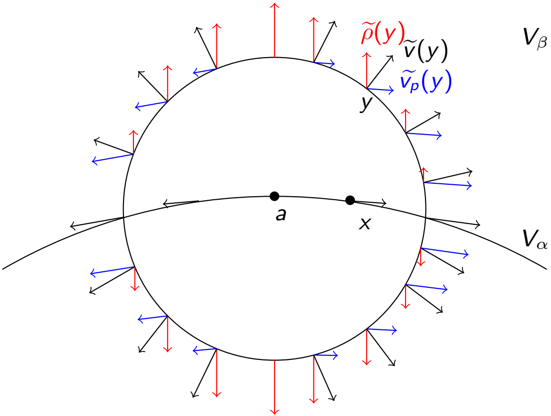
If $v^{(k)}$ is an r-frame tangent to the stratum V_α, then its Lipschitz parallel extension is a Lipschitz stratified r-frame in a neighbourhood of V_α.

Proposition (Result B)

For each stratum $V\alpha_0$, there is a neighbourhood of V_{α_0} and a transverse vector field which is a stratified Lipschitz vector field.

The (local) radial extension of a vector field v is the vector field defined in $\mathcal{T}_\varepsilon(U_\alpha)$ by:

$$\widetilde{v}(y) = \widetilde{v_p}(y) + \widetilde{\rho}(y)$$



The local radial extension of r-frames - The induction.

Consider a stratum V_α with (complex) dimension s_α.
Suppose we know a stratified r-frame $v^{(r)} = (v^{(r-1)}, v_r)$, section of $V_r(TM)$ over a $2q_\alpha$-cell in $(D)^{2p} \cap V_\alpha$ (where $q_\alpha = s_\alpha - r + 1$), with an isolated singularity (zero of v_r).

For a sufficiently small neighbourhood U_α of V_α, we define in $(D)^{2p} \cap U_\alpha$
- the parallel extension of $v^{(r)}$: $(\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p)$, using Result A.
- the transverse vector field: $\widetilde{\rho}$, using Result B.

Proposition (Local radial extension of a frame - M.-H. Schwartz using Whitney, B. - Mostowski - Thuy using Lipschitz)

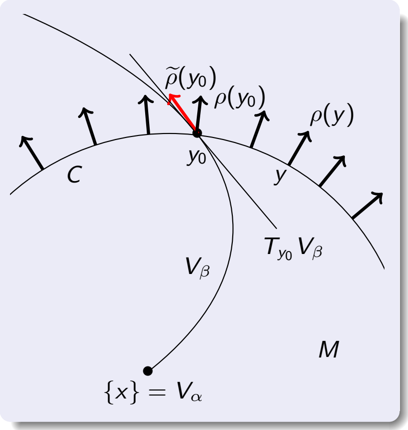
Let d^{2p} a $2p$-cell in $(D)^{2p} \cap U_\alpha$. The radial extension of $v^{(r)}$ in d^{2p}, defined by

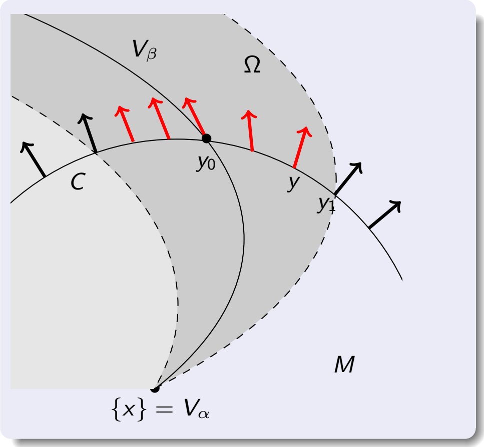
$$\widetilde{v}^{(r)} = (\widetilde{v}_p^{(r-1)}, (\widetilde{v}_r)_p + \widetilde{\rho})$$

satisfies the following conditions:
- the $(r-1)$-frame $\widetilde{v}_p^{(r-1)}$ has no singularity on d^{2p}.
- if $v^{(r)}$ admits an isolated singularity at the barycenter $a \in d^{2p} \cap V_\alpha$ which is a zero of v_r, then $\widetilde{v}^{(r)}$ satisfies the same properties in d^{2p}.
- the index of the extension $\widetilde{v}^{(r)}$ at a, considered as an r-frame tangent to M (in d^{2p}) is equal to the index of $v^{(r)}$ at a considered as an r-frame tangent to V_α.

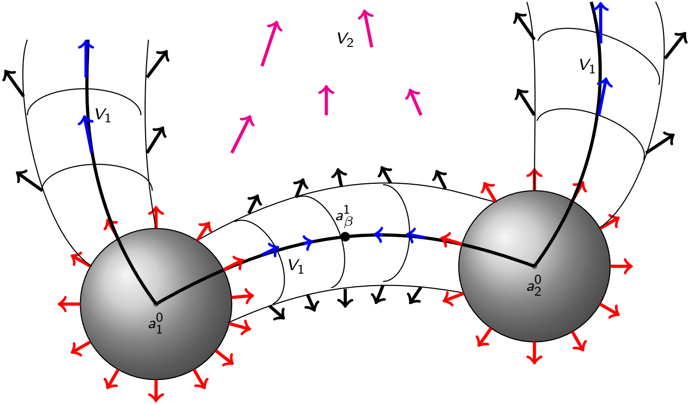
The difficulties using Whitney.

It is clear that the obtained vector fields $\widetilde{v}_p(y)$ and $\widetilde{\rho}(y)$ are not necessarily continuous, as vector fields tangent to M. Indeed, let us look at the case of the field $\widetilde{\rho}(y)$.





Global radial extension of a frame.



The global construction- case $r = 1$.
The radial vector field is constructed in the order : red, blue, black, magenta.

Theorem (M.-H. Schwartz, 1965 (by Whitney stratifications), B. - Mostowski - Thuy 2024 (by Lipschitz stratifications))

One can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)} = (v^{(r-1)}, v_r)$, called radial frame, whose singularities satisfy the following properties:

- *$v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r.*
  On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point.
  On $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.

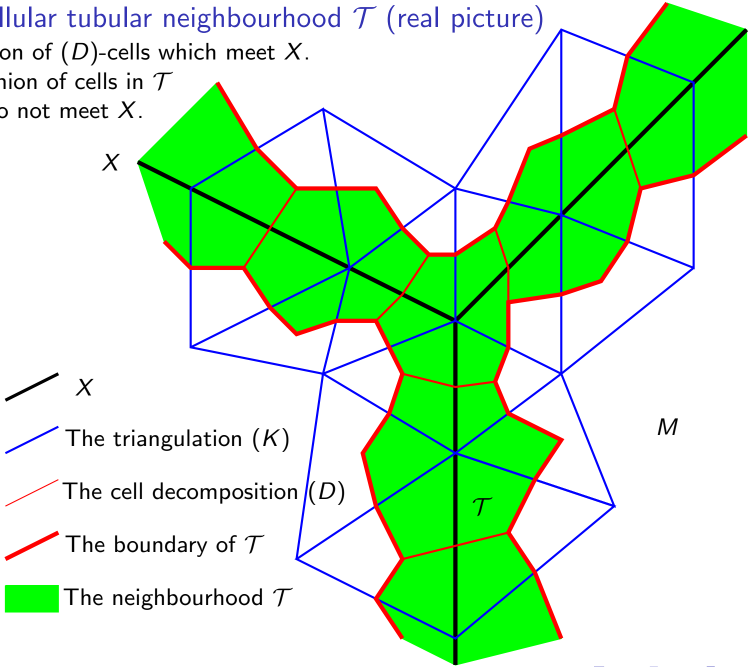
- *Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s_\alpha$-dimensional stratum V_α.*
  If $s_\alpha = r-1$, then $I(v^{(r)}, a) = +1$.
  If $s_\alpha > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_\alpha \cap (D)^{2p}$ considered as an r-frame tangent to V_α.

$$I(v^{(r)}, a; (D)^{2p}) = I(v^{(r)}, a; V_\alpha \cap (D)^{2p}).$$

- *The r-frame $v^{(r)}$ has no singularity on the $2p$-cells of the boundary of a cellular tubular neighbourhood $\mathcal{T}$ of X.*

The cellular tubular neighbourhood $\mathcal{T}$ (real picture)

$\mathcal{T}$ is union of (D)-cells which meet X.
$\partial\mathcal{T}$ is union of cells in $\mathcal{T}$ which do not meet X.



Obstruction cocycles and classes

Denote by $\widehat{d}$ the barycenter of the $2p$-cell d.
Denote by d^* the elementary (D)-cochain whose value is 1 at the $2p$-cell d and 0 at all other cells.

Define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\widetilde{c} : d \mapsto I(v^{(r)}, \widehat{d}) \qquad \widetilde{c} = \sum_{\substack{d \in \mathcal{T} \\ \dim d = 2p}} I(v^{(r)}, \widehat{d})\, d^* \,.$$

The cochain $\widetilde{c}$ actually is a cocycle. Its class $c^p(X)$ lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction along the rays of $\mathcal{T}$ and the second by excision (by $M \setminus \mathcal{T}$).

Definition [M.-H. Schwartz, March 22, 1965]
The p-th Schwartz class of $X \subset M$ is the class

$$c_S^p(X) \in H^{2p}(M, M \setminus X).$$

Deligne-Grothendieck conjecture

The following result was conjectured by Deligne and Grothendieck in 1969 in the framework of algebraic complex varieties.

Let $\mathcal{F}$ be the covariant functor of constructible functions and let $H_*(\ ;\mathbb{Z})$ be the usual covariant $\mathbb{Z}$-homology functor. Then there exists a unique natural transformation

$$c_* : \mathcal{F} \to H_*(\ ;\mathbb{Z})$$

satisfying $c_*(1_X) = c^*(X) \cap [X]$ if X is a manifold.

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The conjecture means that for every algebraic complex variety, one has a functor $c_* : \mathcal{F}(X) \to H_*(X;\mathbb{Z})$ satisfying the following properties:

- $c_*(\varphi + \psi) = c_*(\varphi) + c_*(\psi)$ for φ and ψ in $\mathcal{F}(X)$,
- $c_*(f_*\varphi) = f_*(c_*(\varphi))$ for $f : X \to Y$ morphism of algebraic varieties and $\varphi \in \mathcal{F}(X)$,
- $c_*(1_X) = c^*(X) \cap [X]$ if X is a manifold.

MacPherson classes

In 1974, MacPherson defines a transformation $c_* : \mathcal{F} \to H_*(\ ; \mathbb{Z})$ satisfying the Deligne-Grothendieck conjecture. He uses two ingredients :

1. local Euler obstruction,
2. Mather classes (aleready defined by Wu Wen Tsun in 1965).

Definition (R. MacPherson)

The MacPherson class of X is defined by $c_M(X) = c_M(1_X)$.

Theorem (J.-P. B. – M.-H. Schwartz (1981))

The MacPherson class $c_M(X) = c_M(1_X)$ *is the image of the Schwartz class* $c_S(X)$ *by the Alexander duality isomorphism*

$$H^{2p}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

Definition

Nowadays, these classes are named Chern-Schwartz-MacPherson classes or simply Schwartz-MacPherson classes and are denoted by

$$c_{SM}(X).$$

Dziękuję bardzo za uwagę

Thanks a lot for your attention

Merci pour votre attention

Happy birthday, Wiesław !