Surjectivity of completion for C^∞-rings.
(The algebraic version of Whitney extension theorem)

Dmitry Kerner

Krakow, 26.06.2025

W. Pawłucki's birthday

Plan of the talk:

- Borel lemma, $0 \to (x)^\infty \to C^\infty(\mathbb{R}^n) \to \widehat{C^\infty(\mathbb{R}^n)} \to 0$.
- Functions with "prescribed derivatives of finite and transfinite order" at one point.
- Ideals $I \subset C^\infty(\mathcal{U})$ and "ghost ideals".
- Functions with "prescribed derivatives of finite and transfinite order" over a closed subset $Z \subset \mathcal{U}$.

Joint work with Genrich Belitskii (BGU).

$C^\infty(\mathbb{R}^n) \to \mathbb{R}[\![x]\!], \quad f \to Taylor_o(f).$

$C^\infty(\mathbb{R}^n) \to \mathbb{R}[\![x]\!]$, $f \to Taylor_o(f)$. What is the image of this map?

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A reformulation:

$$f \in C^\infty(\mathbb{R}^n)$$

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$f \in C^\infty(\mathbb{R}^n)$

$jet_0(f)$

$\Cap$

$C^\infty(\mathbb{R}^n)/(x)$

$C^\infty(\mathbb{R}^n) \to \mathbb{R}[\![x]\!]$, $f \to Taylor_o(f)$. What is the image of this map?

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A reformulation:

$$\begin{tikzcd}
 & & f \in C^\infty(\mathbb{R}^n) \arrow[dll] \arrow[dl] \\
jet_0(f) \arrow[d, phantom, "\cap" description] & jet_1(f) \arrow[l] \arrow[d, phantom, "\cap" description] & \\
C^\infty(\mathbb{R}^n)/(x) & C^\infty(\mathbb{R}^n)/(x)^2 \arrow[l] &
\end{tikzcd}$$

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\Cap & \Cap & & \Cap & & \\
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 & & & f \in C^\infty(\mathbb{R}^n) \arrow[dlll] \arrow[dll] \arrow[d] & \\
jet_0(f) \arrow[d, phantom, "\pitchfork"] & jet_1(f) \arrow[l] \arrow[d, phantom, "\pitchfork"] & \cdots \arrow[l] & jet_d(f) \arrow[l] \arrow[d, phantom, "\pitchfork"] & \cdots \arrow[l] \leftarrow \hat{f} \in \underset{\substack{\longleftarrow \\ proj.limit}}{\lim} C^\infty(\mathbb{R}^n)/_{(x)^\bullet} =: \underbrace{\widehat{C^\infty(\mathbb{R}^n)}}_{the\ completion} = \mathbb{R}[\![x]\!] \\
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$$\begin{array}{ccccccc} & & & f \in C^\infty(\mathbb{R}^n) & & & \\ & \swarrow & \swarrow & \downarrow & & \searrow & \\ jet_0(f) & \leftarrow & jet_1(f) \leftarrow \cdots \leftarrow & jet_d(f) & \leftarrow \cdots \leftarrow & \hat{f} \in \underbrace{\varprojlim C^\infty(\mathbb{R}^n)/(x)^\bullet}_{proj.limit} & =: \underbrace{\widehat{C^\infty(\mathbb{R}^n)}}_{the\ completion} = \mathbb{R}[\![x]\!] \\ \cap & & \cap & \cap & & & \\ C^\infty(\mathbb{R}^n)/(x) & \leftarrow & C^\infty(\mathbb{R}^n)/(x)^2 \leftarrow \cdots & C^\infty(\mathbb{R}^n)/(x)^{d+1} & & & \end{array}$$

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 & & & f \in C^\infty(\mathbb{R}^n) \arrow[dlll] \arrow[dll] \arrow[d] \arrow[drr, blue] & & \\
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\end{tikzcd}$$

Then: $0 \to (x)^\infty \to C^\infty(\mathbb{R}^n) \overset{Borel}{\twoheadrightarrow} \widehat{C^\infty(\mathbb{R}^n)} \to 0.$

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$$\underset{\substack{\longleftarrow \\ proj.limit}}{\lim} C^\infty(\mathbb{R}^n)/(x)^\bullet =: \underbrace{\widehat{C^\infty(\mathbb{R}^n)}}_{the\ completion} = \mathbb{R}[\![x]\!]$$

Then: $0 \to (x)^\infty \to C^\infty(\mathbb{R}^n) \overset{Borel}{\twoheadrightarrow} \widehat{C^\infty(\mathbb{R}^n)} \to 0$.

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Why is this useful? (approximate solutions)

$C^\infty(\mathbb{R}^n) \to \mathbb{R}[\![x]\!]$, $f \to Taylor_o(f)$. What is the image of this map?

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Why is this useful? (approximate solutions) Given a problem, let's solve it $mod(x)^2$,

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\cap & & \cap & & \cap & & & & \\
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Why is this useful? (approximate solutions) Given a problem, let's solve it $mod(x)^2$, $mod(x)^3$, ...
Borel: *There exists a "C^∞-solution mod (flat functions)".*

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\Cap & \Cap & & \Cap & & \\
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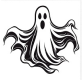

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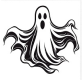

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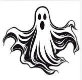

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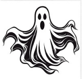

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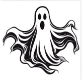

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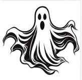

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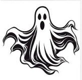

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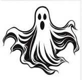

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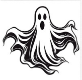

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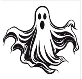

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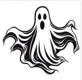

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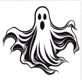

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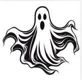

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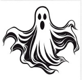

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Corollary (of Whitney extension theorem): $C^\omega(\mathcal{U} \setminus Z) \cap C^\infty(\mathcal{U}) \twoheadrightarrow \widehat{C^\infty(\mathcal{U})}$.

In words: suppose some problem is resolvable on Z (i.e. *mod* $I(Z)$), and also *mod* $I(Z)^2$, and . . . .
Then there exists a C^ω/C^∞-solution *mod* $I(Z)^\infty$.

This Corollary is immediate from Whitney extension theorem.

In various applications we meet ideals not of type $I(Z)^\bullet$.

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Example 1.

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Example 1. Let X be $\mathbb{R}/\mathbb{C}$-analytic, with some stratification $X \setminus Z_0$, $Z_0 \setminus Z_1$, $Z_1 \setminus Z_2$, ... Then the relevant ideals are $I(Z_0)^{p_0}$

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Let $Z_{closed} \subset \mathcal{U}_{open} \subseteq \mathbb{R}^n$. Take $I(Z) \subset C^\infty(\mathcal{U})$. Get the filtration $C^\infty(\mathcal{U}) \supset I(Z) \supset I(Z)^2 \supset \cdots$.

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2. Suppose at first iteration you resolve the problem over $Z_1 \subset \mathcal{U}$,

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Roughly: if a problem is resolvable *mod* I_1, *mod* I_2, ..., then it is "C^ω/C^∞-resolvable *mod* I_∞."

What means $I_\bullet\sim I_\infty+I_\bullet\cap I(Z)^\bullet$?

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- And so on. For each d: $V_d(I_\bullet)$ stabilize.
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Remarks. 1.The proof (a mixture of Algebra and Analysis) does not follow from Whitney extension theorem.
2. This version is for $C^\infty(\overline{\mathcal{U}})$. There are versions for $C^\infty(\mathcal{U})$, $C^\infty(\mathbb{R}^n,o)$, $C^\infty(\mathbb{R}^n,Z)$, $C^\infty(\dots)/_J$.

The general case.

Let $\mathcal{U} \subset \mathbb{R}^n$ be open, bounded, so that $\overline{\mathcal{U}}$ is compact. Take a filtration $C^\infty(\overline{\mathcal{U}}) \supset I_1 \supset I_2 \supset I_3 \supset \cdots$.
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