Jumps of the Milnor numbers in linear deformations of plane curve singularities

Tadeusz Krasiński
(joint results with A. Zakrzewska)
University of Łódź

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Introduction

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Deformation of f_0

$$f_s(x, y)$$

G.M. Greuel, C.Lossen, E.Shustin „Introduction to Singularities and Deformations" 2007.

Introduction

Adjacency problem: when a singularity (or a class of singularities) can be deformed to another one (class of singularities)?

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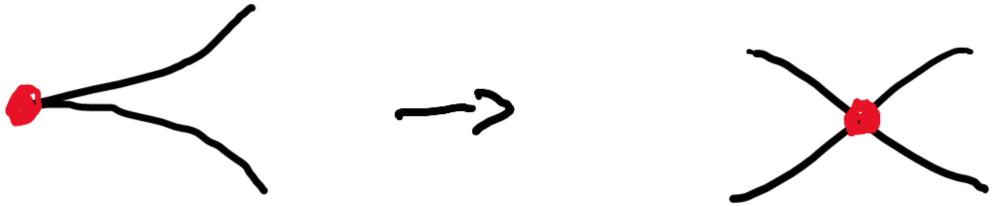
Example.

$$A_2 \to A_1, \qquad x^3 + y^2 \;\to\; x^3 + y^2 + sx^2$$



A_2-singularity $\qquad$ A_1-singularity

Introduction

Example.

$$X_9 \qquad \to \qquad E_7,$$

$$x^4 + y^4 + ax^2y^2 \qquad \to \qquad y(x^3 + y^2)$$

Deformation:

$$x^4 + y^4 + ax^2y^2 + s(x + \alpha y)^3$$

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Consider a specific element of the class X_9

$$f_0 = x^4 + y^4, \qquad \mu(f_0) = 9$$

Fact: It is impossible to deform f_0 in such a way that

$\mu(f_s) = 8$ for $s \neq 0$.

There exist deformations for which $\mu(f_s) = 7$ for $s \neq 0$:

$$f_s = x^4 + (y^2 + sx)^2$$

Definition of the jump

In general for similar singularities of the type

$$f_0 = x^n + y^n, \qquad \mu(f_0) = (n-1)^2$$

we can only get deformations (f_s) for which maximally

$$\mu(f_s) = (n-1)^2 - [\frac{n}{2}] \text{ for } s \neq 0$$

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Definition. The jump (of the Milnor numbers) of f_0

$$\lambda(f_0) := \min(\mu(f_0) - \mu(f_s)) \in \mathbb{N}$$

over all deformations (f_s) of f_0 for which $\mu(f_0) - \mu(f_s) \neq 0$.

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Example. $\lambda(x^n + y^n) = [\frac{n}{2}]$.

Definition of the jump

Problem. Give a formula for $\lambda(f_0)$.

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More general problem.

Problem (A'Campo). Compute the sequence of all possible Milnor numbers in deformations of f_0.

$$\mathcal{M}(f_0) = (\mu_0(f_0), \mu_1(f_0), \dots, \mu_k(f_0))$$

$$\mu(f_0) = \mu_0(f_0) > \mu_1(f_0) > \cdots > \mu_k(f_0) = 0$$

Definition of the jump

Fact. $\lambda(f_0)$ is not a topological invariant.

Example. The class $W_{1,0}$: $x^4 + y^6 + (a + by)x^2y^3$.

All singularities $f^{a,b}$ are topologically equivalent.

$$\mu\left(f^{a,b}\right) = 15.$$

$$\lambda\left(f^{0,b}\right) = \lambda(x^4 + y^6 + bx^2y^4) = 1.$$

$\lambda\left(f^{a,b}\right) = \lambda(x^4 + y^6 + ax^2y^3 + bx^2y^4) > 1$ for generic a, b

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Problems have been considered by putting restrictions on:

1. singularities (homog., quasihomog., irreducible,...),
2. class of deformations (non-degenerate, linear,...).

Results.

The results concern the jump for the class of linear deformations

$$f_s = f_0 + sg, \qquad g(0,0) = 0.$$

In this case we have a particular jump

$$\lambda^{lin}(f_0) := \min(\mu(f_0) - \mu(f_s)) \in \mathbb{N}$$

over all **linear** deformations (f_s) of f_0 for which $\mu(f_0) - \mu(f_s) \neq 0$.

Results.

Two powerful tools:

1. **M. Caramiñana, J. Roe** (2007). Necessary and sufficient numerical condition on the Enriques diagrams $E(f_s)$ and $E(f_0)$ so that f_s is a linear deformation of f_0 (recall Enriques diagrams represent topological types of singularities).

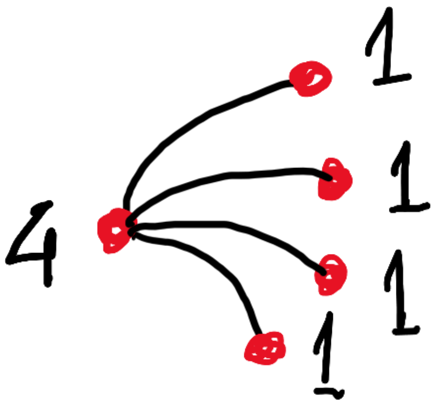
Results.

2. Enriques, Plücker (very long time ago). Formula for the Milnor numer (as intersection multiplicity $i(\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y})$) in terms of Enriques diagrams.

$$\mu(f_0) = \sum_{P \in V^\infty(f_0)} e_P(f_0)(e_P(f_0) - 1) + 1 - r$$

Examples. X_9

$f_0 = x^4 + y^4$



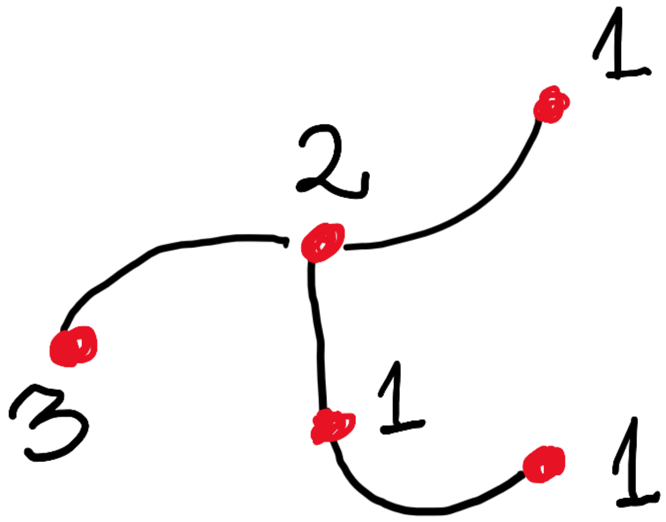
$\mu(f_0) = 4 \cdot 3 + 1 - 4 = 9$

Results.

Examples. E_7

$f_0 = x(x^2 + y^3)$

$\mu(f_0) = 3 \cdot 2 + 2 \cdot 1 + 1 + 1 - 2 = 7$



Results.

Jumps (for linear deformations) of quasihomogeneous singularities.

A.Zakrzewska (2025). If $f_0 = a_{p,0}x^p + \cdots + a_{0,q}y^q$ is a generic quasihomogeneous singularity and $3 \leq p \leq q$

$$\lambda^{lin}(f_0) = \begin{cases} p-2 \; if \; p = q & x^4 + y^4 \\ p-1 \; if \; p \neq q \; and \; p|q & x^3+y^6 \\ \text{GCD}(p,q) \; if \; p \neq q \; and \; p \nmid q & x^4+y^6 \end{cases}$$

More precise results in the paper (for any coefficients).

Results.

Example. For $f_0 = x^5 + y^5$, $\mu(f_0) = 16$, we get

$$\lambda(f_0) = 2,$$

$$\lambda^{lin}(f_0) = 3,$$

$$\lambda^{nd}(f_0) = 4.$$

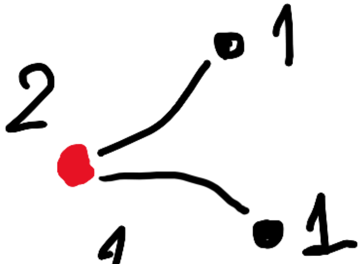
Results.

Characterization of singularities for which $\lambda^{lin}(f_0) = 1$

T.K., A.Zakrzewska (not yet published).

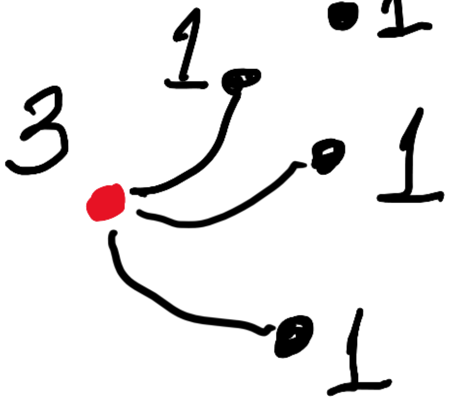
Theorem. $\lambda^{lin}(f_0) = 1$ if and only if the Enriques diagram $E(f_0)$ of f_0 is (we draw Enriques diagrams without leaves):

1. 

(complete E. diagram 

)

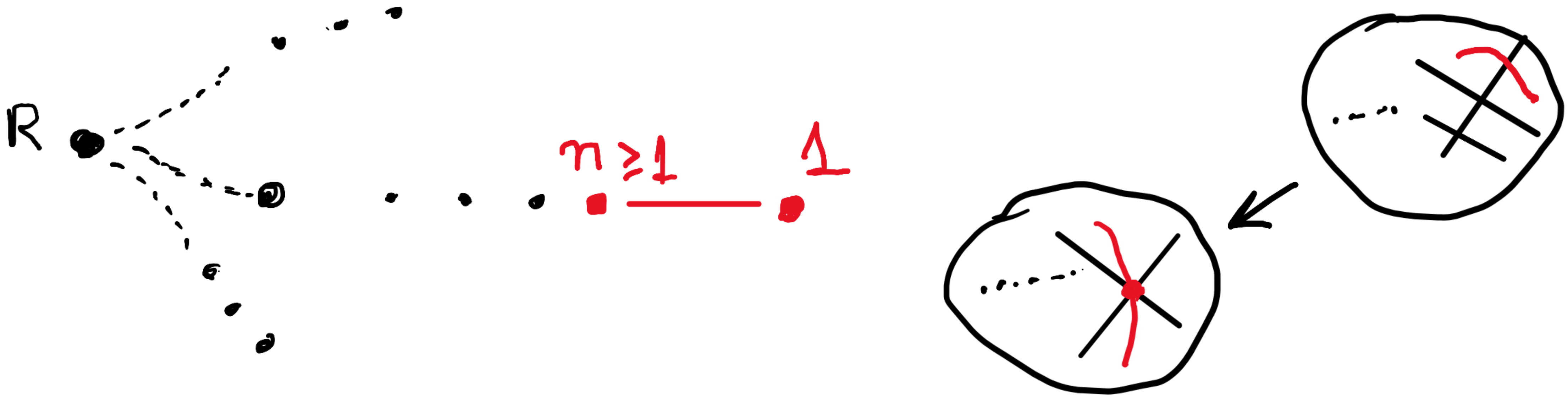
2. 

(complete E. diagram 

)

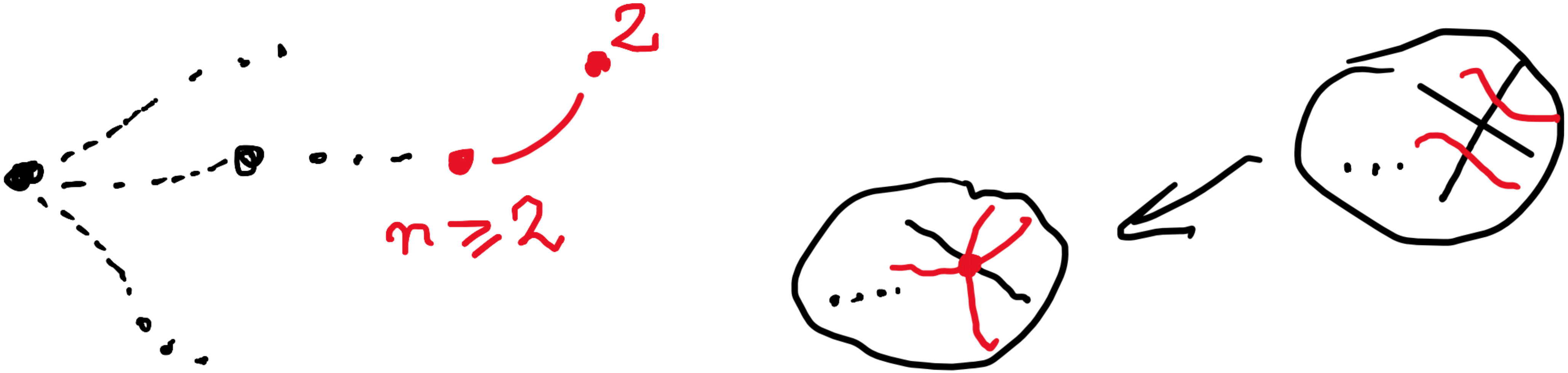
Results.

3.



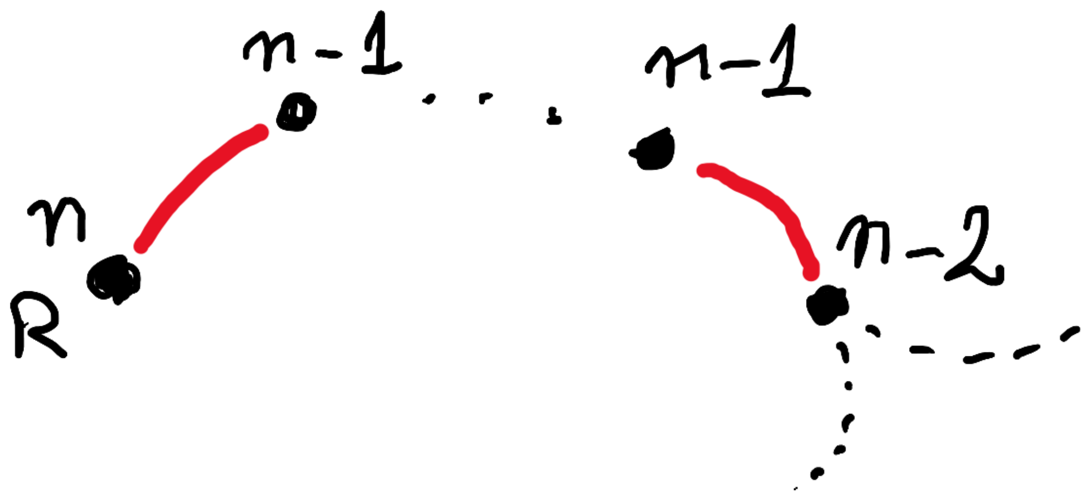
(all irreducible singularities satisfy this condition).

4.

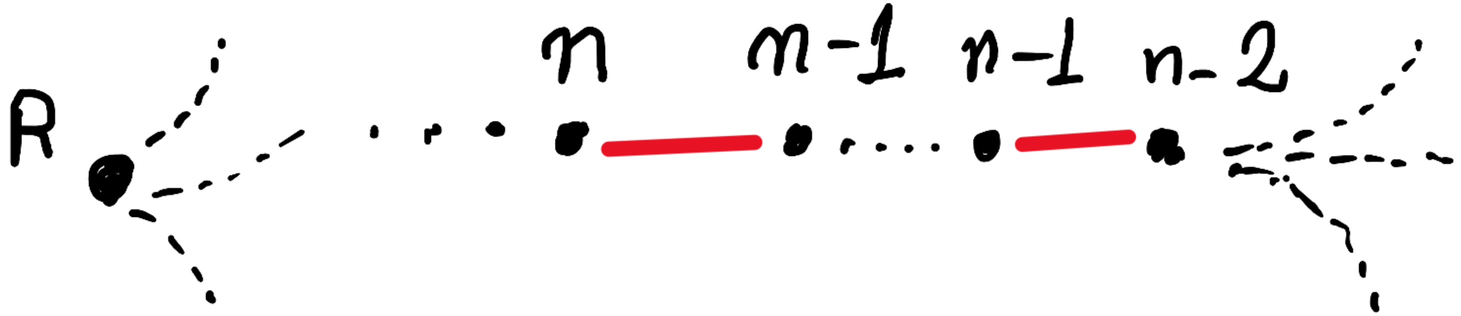


Results.

5.



6.



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3. The first four conditions were discovered by S. Gusein-Zade (1993). We have found only the last two with very specific configuration of orders and branches.
4. The method of proof is elementary and is based on the Caramiñana and Roe result.

Applications to δ-constant deformations.

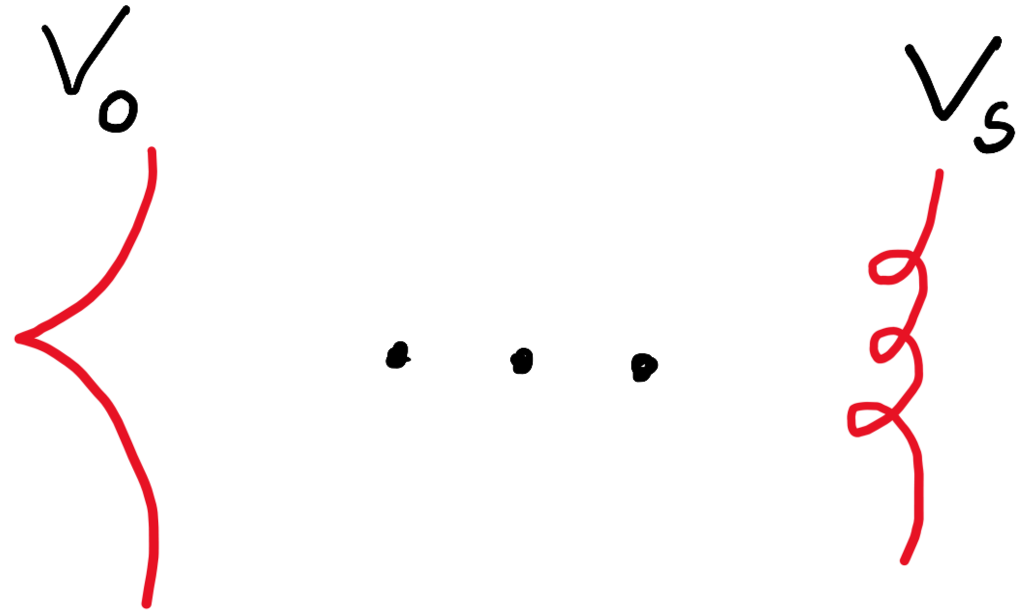
Another discrete invariant of a singularity f_0 is δ-invariant (called also Hironaka number). It is a topological invariant given by the Milnor Formula

$$2\delta = \mu + r - 1,$$

where μ is the Milnor number of f_0 and r the number of branches of $V(f_0)$, or geometrically as the maximal number of singularities in deformations of $V(f_0)$. Precisely

Applications to δ-constant deformations.

Take a generic deformation (V_s) of the zero set $V_0 = V(f_0)$ of f_0 which has only A_1-singularities. Their number is $\delta(f_0)$.



Since A_1-singularities are double ordinary singular points, $\delta(f_0)$ is also called the **number of double points** of $V(f_0)$.

Applications to δ-constant deformations.

Problem. Describe singularities for which there exist δ-constant linear deformations and if yes give topological types in such deformations.

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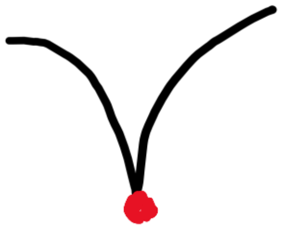
Remark. In any μ-constant deformations elements have the same topological type.

Applications to δ-constant deformations.

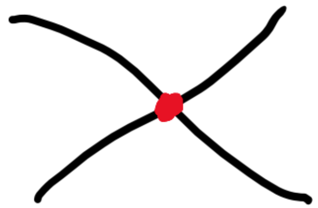
Example. $f_s(x,y) = x^2 + y^3 + sy^2$

$s = 0$



$\mu = 2$
$\delta = 1$

$s \neq 0$



$\mu = 1$
$\delta = 1$

In this δ-constant linear deformation the topological type changes.

Applications to δ-constant deformations.

Using the Milnor formula in any deformation (f_s)

$$2\delta_s = \mu_s + r_s - 1$$

we see the deformation (f_s) is δ-constant if and only if the jump of the Milnor number is equal to the jump up (increase) of the number of branches.

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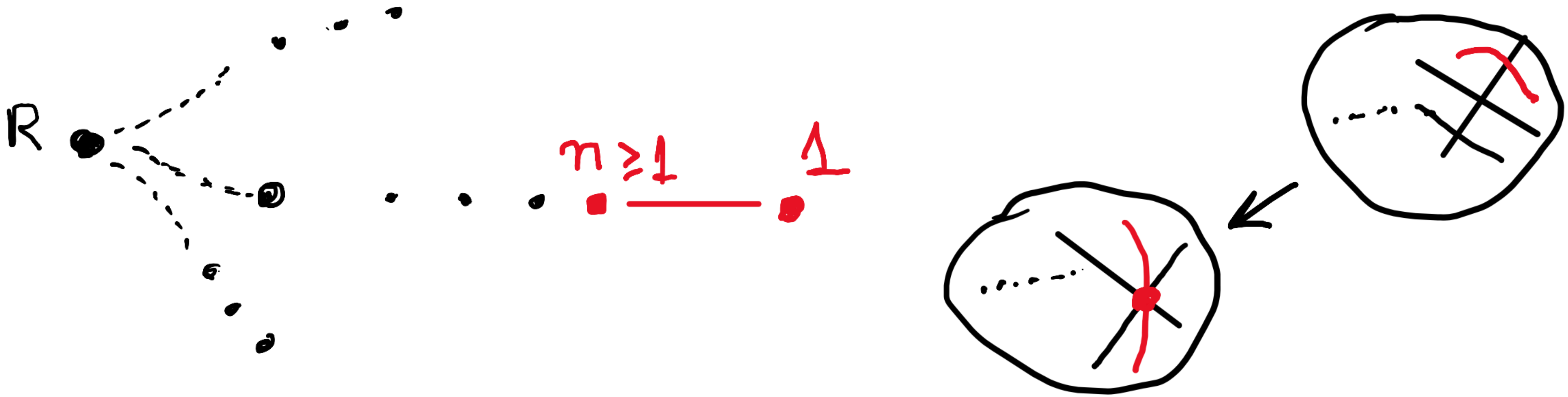
By the theorem we know topological types of singularities and its linear deformations for which the jump of the Milnor numbers is equal to 1. So, it is easy to check in which of these cases the jump of number of branches is also equal to 1. Hence we get

Applications to δ-constant deformations.

Theorem. A singularity f_0 has a δ-constant linear deformation (f_s) with the jump of the Milnor numbers equal to 1 if and only if it has type 3 in the Theorem.

Applications to δ-constant deformations.

3.

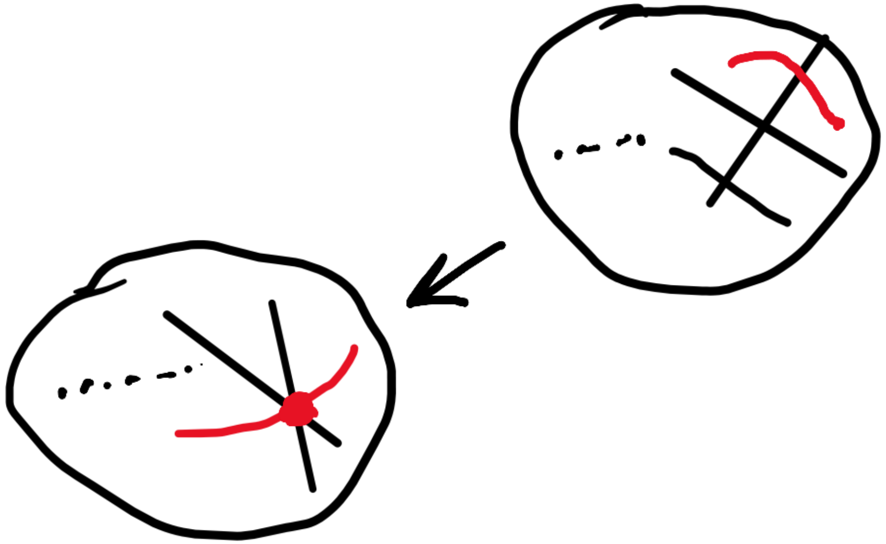


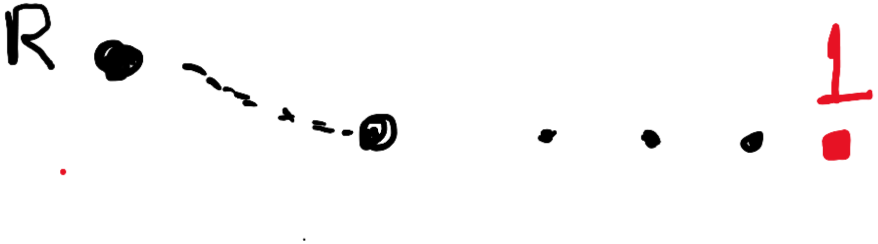
(all irreducible singularities satisfy this condition).

Applications to δ-constant deformations.

Corollary. If f_0 is an ireducible singularity the there exists a δ-constant linear deformation of f_0 and its toplogical type is given by the below Enriques diagram. If the jump of this deformation is 1 then this topological type is unique.

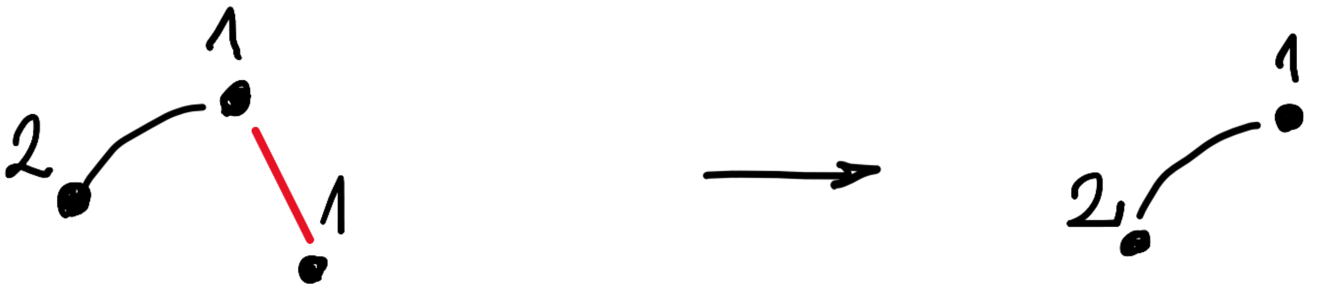
Applications to δ-constant deformations.





Applications to δ-constant deformations.

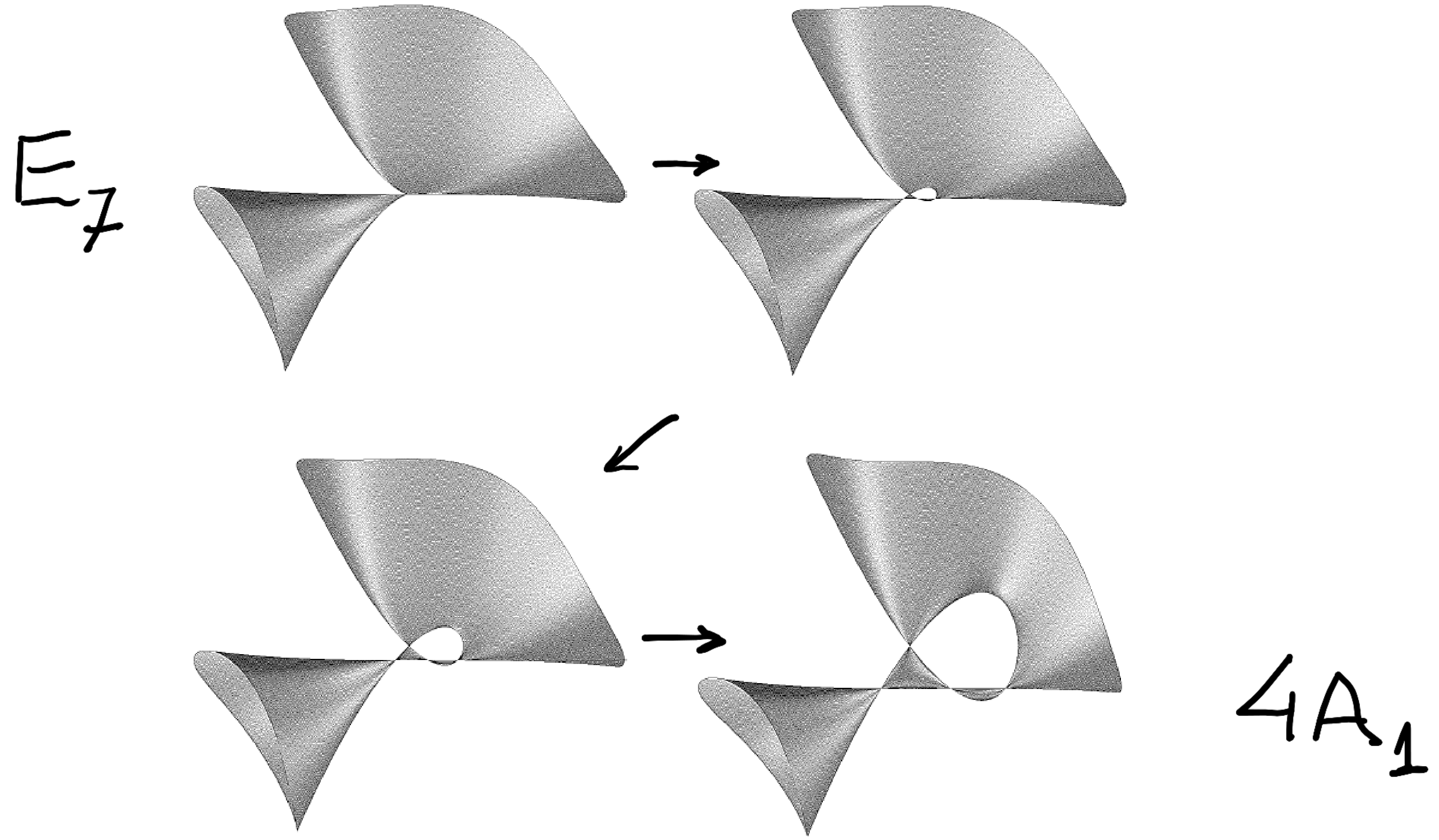
Example. Take the cusp $f_0(x,y) = x^2 + y^3$



For complete Enriques diagrams

$$f_s(x,y) = x^2 + y^3 + sy^2$$

Thank you.



Picture of a deformation from the monograph ”Introduction to Singularities and deformations” by Greuel, Lossen and Shustin.