Strong stratifications and term description over valued fields with analytic structure, applications

Conference in honour of
Wiesław Pawłucki's 70th birthday

Tame geometry and extension of functions
Kraków, 26 June 2025

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Main aim

We establish a certain strong smooth stratification of sets and a term description of functions, which are definable over valued fields K (possibly non algebraically closed) with separated analytic structure. The basic tools are: elimination of valued field quantifiers, term structure of definable functions, Lipschitz cell decomposition with preparation of RV-parametrized sets, and a non-Archimedean definable version of Bierstone–Milman's canonical desingularization algorithm, achieved in an earlier paper of ours. As application, we give uniform Yomdin–Gromov parametrizations over Henselian fields K, whose leading term structure $RV(K)$ is a group with some finite prime invariant.

We deal with geometry over a non-trivially valued field K of equicharacteristic zero with an analytic $\mathcal{A}$-structure over a separated Weierstrass system $\mathcal{A} = \{A_{m,n}\}_{m,n \in \mathbb{N}}$.

Analytic language, quantifier elimination

The field K will be considered with the analytic language $\mathcal{L}_{\mathcal{A}}$. It is a two sorted language: on the main, valued field sort K, the language of rings $(0, 1, +, -, \cdot)$ augmented with the multiplicative inverse $(\cdot)^{-1}$ (with convention $0^{-1} = 0$) and the names of all functions of the collection $\mathcal{A}$; on the auxiliary sort $RV(K)$, the language $\mathcal{L}_{rv}$ introduced in [N5, Section 2]. Power series from $A_{m,n}$ are construed via σ as $f^{\sigma} = \sigma_{m,n}(f)$ on their natural domains, and as zero outside them. Note that the definable sets in the auxiliary sort RV are precisely those already definable in the pure valued field language. Denote by $T_{\mathcal{A}}$ the $\mathcal{L}_{\mathcal{A}}$-theory of all Henselian, non-trivially valued fields of equicharacteristic zero with analytic $\mathcal{A}$-structure.

The theory $T_{\mathcal{A}}$, unlike that of strictly convergent analytic structures, admits elimination of valued field quantifiers in the language $\mathcal{L}_{\mathcal{A}}$ (cf. [CL1, Theorem 6.3.7]).

Resolution of singularities

It seems, however, that the rings $A^{\dagger}_{m,n}(K)$ of global analytic functions with two kinds of variables defined on the spaces

$$(K^{\circ})^m \times (K^{\circ\circ})^n, \quad m, n \in \mathbb{N},$$

suffer from lack of good algebraic properties. Only the rings $A^{\dagger}_{m,0}(K)$ and $A^{\dagger}_{0,n}(K)$ of power series with one kind of variables enjoy good algebraic properties being, as in the classical rigid analytic geometry, Noetherian, factorial, normal and excellent (as they fall under the Weierstrass–Rückert theory; cf. [CL1, BGR]). Therefore the techniques of resolution of singularities, as e.g. the ones by Bierstone–Milman [BM] or Temkin [T], cannot be directly applied to those spaces. The opposite situation holds for strictly convergent analytic structures. There the rings of global analytic functions are excellent, but elimination of valued field quantifiers requires to augment the language with the henselian functions (cf. [CL2, Theorem 3.4.4]).

Desingularization of terms and strong stratification

Theorem

Every $\mathcal{L}_{\mathcal{A}}$-definable subset X of $(K^\circ)^n$ is a finite disjoint union of strong analytic, subanalytic manifolds $S_k := \sigma_k(W_k)$, $k = 1, \dots, s$, such that:
1) Each $\sigma_k : N_k \to (K^\circ)^n$ is a multi-blowup with exceptional divisor E_k; we can assume that N_k are analytic submanifolds of $(K^\circ)^n \times \mathbb{P}^N(K)$ for all k and an integer N large enough.
2) For each $k = 1, \dots, s$, $W_k := (Z_k \cap \Omega_k) \setminus E_k$, where $U_k \subset N_k$ is a clopen subdomain of N_k or of the pre-image of an admissible smooth center C (where a new process starts via induction), Z_k is an analytic subset of U_k and Ω_k is an open definable subset of U_k.
3) The restriction $\sigma_k | W_k : W_k \to S_k$ is a homeomorphic parametrization of the stratum S_k. □

It seems that the only results on stratification known before are the ones for subanalytic subsets of affine spaces over an algebraically closed valued field, provided in [LR2], [LR3], [CL2].

Term description after some algebraic Skolemization

Proposition

The theory $T_{\mathcal{A}}$ has the following property uniformly in its models K. Let $f : X \to K$ be an $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable function on a subset X of K^n. Then f is given piecewise on $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable sets by $\mathcal{L}^{*}_{\mathcal{A}}$-terms, i.e. there exist a finite $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable partition $X = X_1 \cup \ldots \cup X_s$ and $\mathcal{L}^{*}_{\mathcal{A}}$-terms $t_1, \ldots, t_s$ such that $f(x) = t_i(x)$ for all $x \in X_i$, $i = 1, \ldots, s$.

Theorem

For every $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable set $X \subset K^n$, there exists a finite decomposition of X into $\mathcal{L}^{\dagger}_{\mathcal{A}}$-definable ordinary cells C_k, $k = 1, \ldots, q$, with centers given by $\mathcal{L}^{*}_{\mathcal{A}}$-terms.
Furthermore, one can additionally require that each C_k be, after some coordinate permutation, a cell of type $(1, \ldots, 1, 0, \ldots, 0)$ with 1-Lipschitz centers $c_k = (c_{k,1}, \ldots, c_{k,n})$, $k = 1, \ldots, q$.

Yomdin–Gromov parametrizations

We shall prove that definable subsets of $\mathcal{O}_K^n$ admit, uniformly in definable families, certain Yomdin–Gromov T^r-parametrizations.

The concept of Yomdin–Gromov parametrization goes back to their papers [Yo, Gro]. The crucial result, the Yomdin–Gromov algebraic lemma, provides also a polynomial estimate for the number of semialgebraic $\mathcal{C}^r$-parametrizations of a given bounded semialgebraic subset X of $\mathbb{R}^n$.

Its generalizations to the o-minimal settings (with remarkable applications in the fields of dynamics and Diophantine geometry), were initiated by Pila–Wilkie [PW], allowing them to establish their counting theorem about rational points on definable sets.

Some other approaches have been recently given in the papers [CPV, BN1, BN2, CPW]. Note also that the notion of complexity is not available in the general o-minimal settings, and must be replaced by uniformity over families.

Basic terminology

For any valued field K, a subset P of $\mathcal{O}_K^m$ and a positive integer r, a function $f = (f_1, \ldots, f_n) : P \to \mathcal{O}_K^n$ is said to satisfy *T_r-approximation* if P is open in $\mathcal{O}_K^m$, and for each $a \in P$ there is an n-tuple $T_{f,a}^{<r}$ of polynomials with coefficients in $\mathcal{O}_K$ of degree $< r$ such that $\left| f(x) - T_{f,a}^{<r}(x) \right| < |x - a|^r$ for all $x \in P$.

Clearly, such polynomials $T_{f,a}^{<r}$ are unique, and if f is of class C^r, they are just the Taylor polynomials of f at a of order r.

Obviously, f satisfies T^1-approximation if and only if f is 1-Lipschitz.

Notice that, T_r-parametrizations will be achieved, as in the papers [CCL, CFL], from T_1-ones by precomposing with power functions, with Cauchy's inequalities playing a key role. Therefore the concept of T^1-parametrization should be strengthened to allow the use of Cauchy's inequalities for the involved terms defined on open polydiscs.

Strong T_1-parametrization

Hence the following two cases, affecting the definition of strong approximation below, are encountered:

(V1) the value group vK is densely ordered, i.e. it has no smallest positive element 1;
(V2) conversely, vK has a smallest positive element 1.

Let K be a model of the theory $T_{\mathcal{A}}$. We say that the function f as above satisfies strong C^1-approximation if P is an open cell with zero centers, f is given on P by Henselian terms which are 1-Lipschitz on P. Moreover, in case (V2), those terms are required to be 1-Lipschitz on the associated box B_{as} to each box B of P. Here B_{as} is the box over the algebraic closure K_{alg} of L defined by the same condition as B over K.

Strong T_1-parametrization

Theorem

Consider an $\mathcal{L}_{\mathcal{A}}$-definable subset X of $\mathcal{O}_K^k \times \mathcal{O}_K^n$, regarded as a definable family

$$X_w = \{x \in K^n : \ (w,x) \in X\}, \quad w \in W \subset K^m.$$

Suppose that the set X_w is of dimension m for each $w \in W$. Then there exist a finite set I and a definable family $f = (f_{w,i})_{(w,i) \in W \times I}$ of definable functions

$$f_{w,i} : P_{w,i} \to X_w \ \text{ with } \ P_{w,i} \subset \mathcal{O}_K^m,$$

such that $(f_{w,i})_{i \in I}$ is a strong T_1-parametrization of X_w for each $w \in W$. Thus the cardinality $s = s(X)$ is independent of $w \in W$.

T_r-parametrization

Theorem

Consider an $\mathcal{L}_{\mathcal{A}}$-definable subset $X \subset K^k \times \mathcal{O}_K^n$, regarded as a definable family

$$X_w = \{x \in K^n : (w, x) \in X\}, \quad w \in W \subset K^m,$$

and a positive integer r. Suppose that the set X_w is of dimension m for each $w \in W$ and that the congruence invariant $b_r = [r]RV(K)$ is finite. Then there exist a positive integer $s = s(X)$, a finite set I of cardinality $\leqslant s \cdot b_r^m$, and a definable family $f = (f_{w,i})_{(w,i) \in W \times I}$ of definable functions

$$f_{w,i} : P_{w,i} \to X_w \quad \text{with} \quad P_{w,i} \subset \mathcal{O}_K^m,$$

such that $(f_{w,i})_{i \in I}$ is a T_r-parametrization of X_w for each $w \in W$.

□

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