Global Complexification of Restricted Log-Exp-Analytic Functions

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Outline

- Structures, o-minimality and definability
- Global complexification
- Restricted log-exp-analytic functions
- A preparation theorem for restricted log-exp-analytic functions
- Main results of my research: **Tamm's theorem** and **complexification** for restricted log-exp-analytic functions
- Open questions

Structures

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(S3) If $A \in M_{n+1}$ then $\pi_n(A) \in M_n$ where $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n)$, denotes the projection on the first n coordinates.

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(S4) M_n contains the semialgebraic subsets of $\mathbb{R}^n$.

O-minimal structures and definability

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- Let $B \subset \mathbb{R}^n$. A function $f : B \to \mathbb{R}^m$ is $\mathcal{M}$-**definable** if its graph $\{(x, f(x)) \mid x \in B\}$ is $\mathcal{M}$-definable.

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- $\mathbb{R}_{\text{exp}}$: The structure generated by the graph of the global real exponential function.
- $\mathbb{R}_{\text{an,exp}}$: The structure generated by all globally subanalytic sets and the graph of the global real exponential function.

$\mathcal{M}$ denotes a fixed o-minimal structure on the reals.
Definable means $\mathcal{M}$-definable.

Complexification

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Definition

We say that $\mathcal{M}$ has **complexification** if every real analytic definable function has locally a definable holomorphic extension.

Complexification

Example

- Let $\mathcal{M}$ be an o-minimal expansion of $\mathbb{R}_{\text{an}}$ (e.g. $\mathbb{R}_{\text{an}}$ or $\mathbb{R}_{\text{an,exp}}$). Then $\mathcal{M}$ has complexification.

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  - Let $x \in \mathbb{R}$ and let V be an open ball in $\mathbb{C}$ around x.
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  - $F|_V$ is not $\mathbb{R}_{\text{exp}}$-definable (Bianconi, 1997).

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$\mathbb{R}_{\text{exp}}$ does not have global complexification.

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(2) *The o-minimal structure $\mathbb{R}_{\mathrm{an}}$ has global complexification.*

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Ideas for the proof of (2):

By a preparation theorem of Lion and Rolin a globally subanalytic function can be piecewise written as

$$a(t) \cdot |x - \theta(t)|^q \cdot v\big((b_i(t)|x - \theta(t)|^{p_i})_{i=1,\ldots,s}\big)$$

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$A = \{(t,z) \in \pi(C) \times \mathbb{C} \mid \alpha(t) < |z - \theta(t)| < \omega(t), \} \setminus] - \infty, \theta(t)[$
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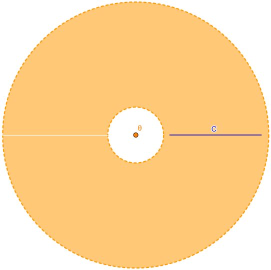

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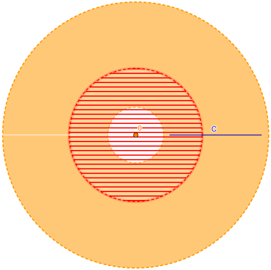

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$\Delta := \{(t,s,z)\in\pi(C)\times\mathbb{R}_{>0}\times\mathbb{C} \mid \alpha(t)<s<\omega(t), z\in B(\theta(t),s/2)\}$
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Finally: Do an induction on the number of variables.

Question:

Does $\mathbb{R}_{\text{an,exp}}$ have global complexification?

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  Examples: x^2 and $\exp(x \cdot y^2 \cdot \log(z))$ are terms.

The univariate case

Theorem (L. van den Dries, A. Macintyre and D. Marker, 1994)

Let $f : \mathbb{R}^n \to \mathbb{R}$ *be definable. Then* f *is piecewise given by terms.*

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Consequence:

Theorem (T. Kaiser, 2016)

Let $U \subset \mathbb{R}$ be open. A definable real analytic function $f : U \to \mathbb{R}$ has a global complexification, i.e. a definable holomorphic extension.

My contribution as a PhD student

I considered definable real analytic functions in more than one variable.

Log-Analytic Functions

Definition

We call a function $f : X \to \mathbb{R}$ **log-analytic** if f is piecewise the composition of globally subanalytic functions and the global logarithm.

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Example

The function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \log(1 + x^2 + |x|),$$

is log-analytic.

The multivariate case

Definition

Let $X \subset \mathbb{R}^n$ be open. We call a function $f : X \to \mathbb{R}$ **restricted log-exp-analytic** if f is the composition of log-analytic functions and exponentials of locally bounded functions.

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The definable function

$$h : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} \exp(-1/x), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

is not restricted log-exp-analytic, but $h|_{\mathbb{R}_{>0}}$ is.

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globally subanalytic $\subset$ **log-analytic** $\subset$ **restricted log-exp-analytic** $\subset$ **log-exp-analytic** $=$ **definable**.

Restricted Log-Exp-Analytic Functions

Results: I established some differentiability results and global complexification for the big class of restricted log-exp-analytic functions.

Restricted Log-Exp-Analytic Functions

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Strategy: Determine a suitable preparation theorem for a restricted log-exp-analytic function as it has been done for globally subanalytic functions.

A Preparation Theorem for Log-Analytic Functions

Definition (Lion/Rolin 1997)

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable. A tuple $(y_0, ..., y_r)$ of functions on C is called **logarithmic scale** on C with **center** $\Theta := (\Theta_0, ..., \Theta_r)$ if the following holds:

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- for $(t, x) \in C$ we have $y_j(t, x) = \log(|y_{j-1}(t, x)|) - \Theta_j(t)$ $(j \in \{1, ..., r\})$.

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In general the partition $\mathcal{C}$ cannot be chosen in this way that $f|_C$ is log-analytically prepared with log-analytic data for every $C \in \mathcal{C}$.

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Consequence: Hard to show that log-analytic functions are closed under global complexification.

A Preparation Theorem for Log-Analytic Functions

Definition

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable. We call $g : \pi(C) \to \mathbb{R}$ C-**heir** if there is a logarithmic scale $(\tilde{y}_0, ..., \tilde{y}_r)$ with center $(\tilde{\Theta}_0, ..., \tilde{\Theta}_r)$ on C and $l \in \{0, ..., r\}$ such that $g = \exp(\tilde{\Theta}_l)$.

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Remark

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be open. A C-nice function is restricted log-exp-analytic since the center of every logarithmic scale on C is locally bounded.

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Theorem (Opris, 2023)

*Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f : X \to \mathbb{R}$ be log-analytic. Then there is a partition $\mathcal{C}$ of X into finitely many definable cells such that $f|_C$ is log-analytically prepared with C-**nice data** for $C \in \mathcal{C}$.*

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This means: For $C \in \mathcal{C}$ we have

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where $a, b_1, \ldots, b_s$ and the center $(\Theta_0, \ldots, \Theta_r)$ of $(y_0, \ldots, y_r)$ are C-nice.

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Remark, Kaiser/Opris 2022

On **simple cells** which are cells of the form

$$C := \{(t, x) \in D \times \mathbb{R} \mid 0 < x < d(t)\}$$

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Consequence: Log-Analytic functions are closed under differentiation.

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$$b_i \cdot |y_0|^{p_{i0}} \cdot \ldots \cdot |y_r|^{p_{ir}} \cdot e^{d_i} \in [-1,1],$$
- $c(t,x)$ and $d_1(t,x)$,...,$d_s(t,x)$ are log-exp-analytically prepared of **lower complexity** than f.

A Preparation Theorem for Restricted Log-Exp-Analytic Functions

Theorem (Opris, 2022)

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and open. Let $f : X \to \mathbb{R}$ be restricted log-exp-analytic. Then there is a partition $\mathcal{C}$ of X into finitely many definable cells such that $f|_C$ is log-exp-analytically prepared with exponentials of functions which are **locally bounded with respect to** X *for every $C \in \mathcal{C}$.*

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where $c, d_1, \ldots, d_s$ are **restrictions of locally bounded functions** on X.

First Result: Tamm's Theorem

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Example

Note that Tamm's theorem does not hold in $\mathbb{R}_{\text{an,exp}}$ in general. Consider

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} e^{-1/x}, & x > 0, \\ 0, & \text{else.} \end{cases}$$

Then f is infinitely often continuously differentiable at 0 but not real analytic.

Tamm's Theorem: Proof Sketch

Lemma (Opris 2022)

Restricted log-exp-analytic functions are closed under taking derivatives.

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Finally: Apply methods of Van den Dries (Gateaux-differentiability) to obtain Tamm's theorem.

First Result: Tamm's Theorem

Corollary

Let $f : X \to \mathbb{R}, (t, x) \mapsto f(t, x)$, *be restricted log-exp-analytic. Then the set of all* $(t, x) \in X$ *such that* $f(t, -)$ *is real analytic at* x *is definable.*

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Corollary

Let $f : X \to \mathbb{R}, (t,x) \mapsto f(t,x)$, be restricted log-exp-analytic. Then the set of all $(t,x) \in X$ such that $f(t,-)$ is real analytic at x is definable.

Note: This does also not hold for definable functions in general.

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Let $f : X \to \mathbb{R}, (t,x) \mapsto f(t,x)$*, be restricted log-exp-analytic. Then the set of all* $(t,x) \in X$ *such that* $f(t,-)$ *is real analytic at* x *is definable.*

Note: This does also not hold for definable functions in general.

Example

The definable function

$$h : \mathbb{R}^2 \to \mathbb{R},\ (t,x) \mapsto \begin{cases} e^{2t\log(|x|)}, & x > 0, \\ 0, & x = 0, \end{cases}$$

satisfies $\{t \in \mathbb{R} \mid h(t,-) \text{ is real analytic at } 0\} = \mathbb{Z}$.

Second Result: Global Complexification

Theorem (Opris, 2022)

Let $X \subset \mathbb{R}^m$ be open and let $f : X \to \mathbb{R}$ be a real analytic restricted log-exp-analytic function. Then f has a global complexification which is again restricted log-exp-analytic.

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Use the preparation theorem for restricted log-exp-analytic functions from above.

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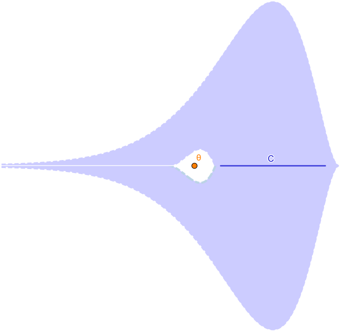

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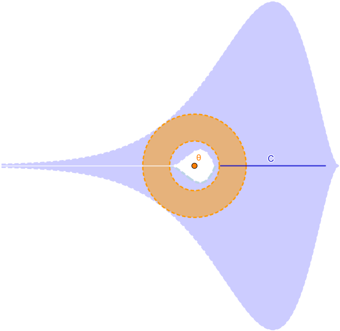

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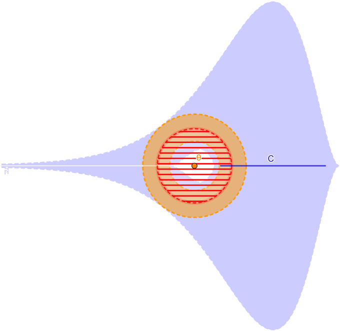

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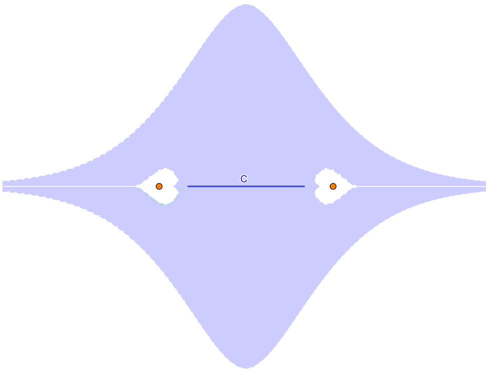

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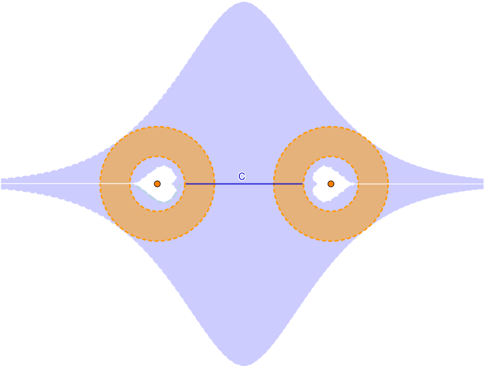

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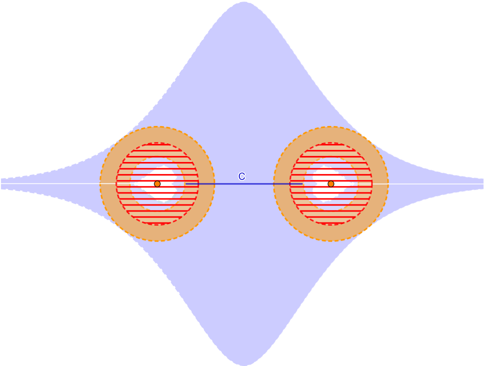

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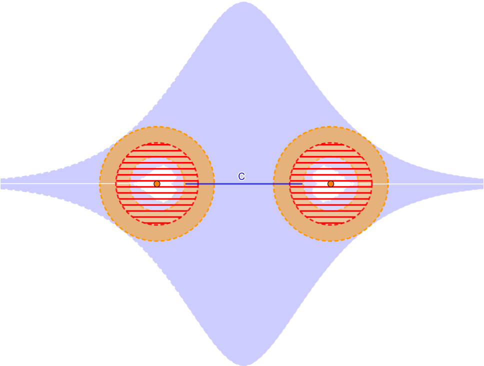

Additional challenges compared to the $\mathbb{R}_{\text{an}}$ case:

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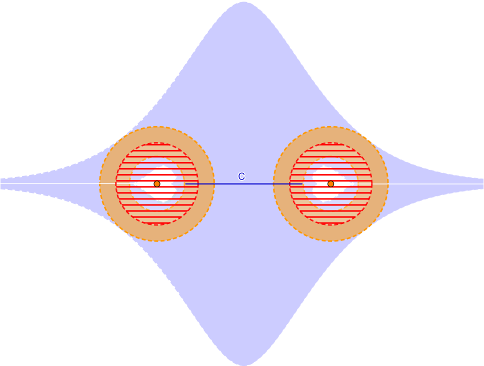

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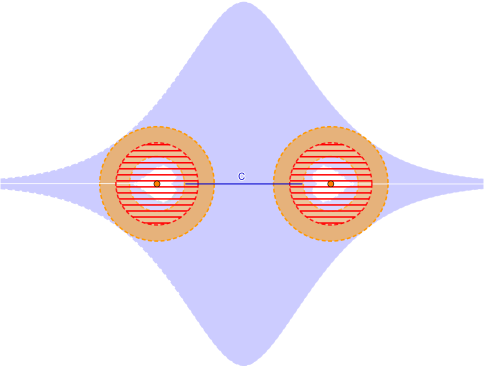

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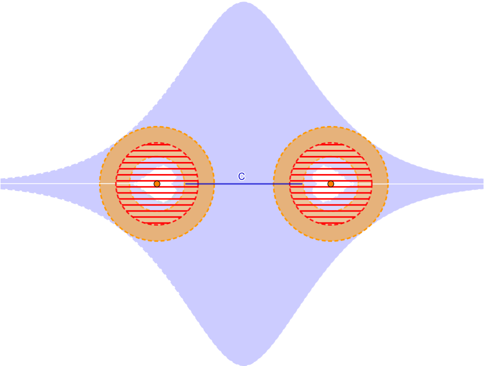

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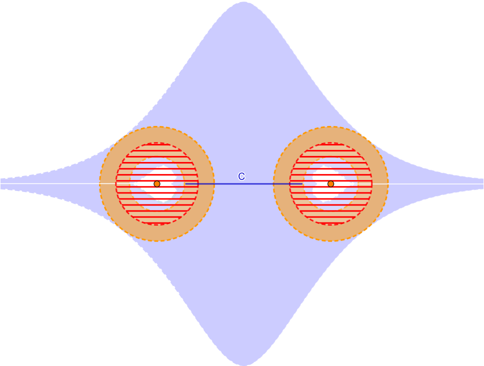

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There is also a **parametric version** of this result.

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The function $f : X \to \mathbb{R}$ where X_t is open for each $t \in \mathbb{R}^n$ is called **restricted log-exp-analytic in** y if f is the composition of log-analytic functions and exponentials of functions g where g_t is locally bounded for every $t \in \mathbb{R}^n$.

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Theorem (Opris, 2022)

Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable such that X_t is open for every $t \in \mathbb{R}^n$. Let $f : X \to \mathbb{R}, (t, y) \mapsto f(t, y)$, be restricted log-exp-analytic and real analytic in y. Then there is a definable $Z \subset \mathbb{R}^n \times \mathbb{C}^m$ with $X \subset Z$ where Z_t is open for every $t \in \mathbb{R}^n$ and $F : Z \to \mathbb{C}, (t, z) \mapsto F(t, z)$, with $F|_X = f$ which is restricted log-exp-analytic in z and F_t is holomorphic for every $t \in \mathbb{R}^n$.

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  - f is flat at zero, but not the zero function.
  - This feature does not occur in the case of real analytic functions.

The publications this talk builds on are

- **Preparation Theorems for $\mathbb{R}_{\mathbf{an,exp}}$ in general**:
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