On the semialgebraic Whitney extension problem

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How can one decide if a function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$ compact, is the restriction of a C^m function $F : \mathbb{R}^n \to \mathbb{R}$?

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- $\mathrm{WEP}_{n,m}$ ✓ [Fefferman 2006]

Bundles

For each $x \in X$ let $H(x) \subseteq \mathcal{P}_n^m = \mathbb{R}[X_1, \ldots, X_n]_{\leq m}$ be empty or an affine subspace (more precisely, a coset of an ideal w.r.t. jet multiplication). In that case, $H(X) := (H(x))_{x \in X}$ is called a bundle. A section of $H(X)$ is a function $F \in C^m(\mathbb{R}^n)$ such that $J_x^m F \in H(x)$ for all $x \in X$.

For example, $H_0(x) := \{P \in \mathcal{P}_n^m : P(x) = f(x)\}$, $x \in X$, defines a bundle $H_0(X)$, and f admits a C^m extension iff there exists a section of $H_0(X)$.

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Glaeser refinement

Given $H(X)$, define its Glaeser refinement $\tilde{H}(X)$: fix a large integer $k = k(m, n)$. For $x_0 \in X$ and $P_0 \in H(x_0)$, we have $P_0 \in \tilde{H}(x_0)$ iff $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1, \ldots, x_k \in X \cap B(x_0, \delta)\ \exists P_1, \ldots, P_k$ with $P_j \in H(x_j)$ and $|\partial^\alpha (P_i - P_j)(x_j)| \leq \varepsilon\, |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \leq m$, $0 \leq i, j \leq k$.

$\tilde{H}(X)$ is a bundle. Each section F of $H(X)$ is also a section of $\tilde{H}(X)$.

We get a sequence of bundles $H_0(X) \supseteq H_1(X) \supseteq \cdots$ which stabilies: $H_\ell(X) = H_{2 \dim \mathcal{P}_n^m + 1}(X) =: H_*(X)$ for $\ell \geq 2 \dim \mathcal{P}_n^m + 1 = 2\binom{n+m}{m} + 1$.

Theorem **[Fefferman 2006]**

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Theorem [Fefferman 2007]

There is a linear bounded extension operator $T : C^m(\mathbb{R}^n)|_X \to C^m(\mathbb{R}^n)$. The norm of T is bounded by a constant depending only on m and n.

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Finiteness principle [Fefferman 2006]

There exist $k = k(m, n)$ and $C = C(m, n)$ such that the following holds. Suppose that $\forall Y \subseteq X$, $\#Y \leq k$, there is an extension $F_Y \in C^m(\mathbb{R}^n)$ of $f|_Y$ with $\|F_Y\|_{C^m(\mathbb{R}^n)} \leq 1$. Then there is an extension $F \in C^m(\mathbb{R}^n)$ of f with $\|F\|_{C^m(\mathbb{R}^n)} \leq C$.

One can take $k = 2^{\dim \mathcal{P}_n^m}$. [Bierstone–Milman 2007]

Semialgebraic sets and functions

The semialgebraic subsets of $\mathbb{R}^n$ are finite unions of sets of the form $\{x \in \mathbb{R}^n : P(x) = 0, Q_1(x) > 0, \ldots, Q_\ell(x) > 0\}$, where $\ell \in \mathbb{N}$ and $P, Q_1, \ldots, Q_\ell \in \mathbb{R}[X_1, \ldots, X_n]$. A map $f : \mathbb{R}^n \supseteq S \to \mathbb{R}^k$ is called semialgebraic if its graph is semialgebraic.

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- $\mathrm{SWEP}_{n,1}$ ✓ [Aschenbrenner–Thamrongthanyalak 2019]; in arbitrary o-minimal expansions of real closed fields, based on a definable version of Michael's selection theorem.

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- $\mathrm{SWEP}_{2,m}$ ✓ [Fefferman–Luli 2022]
- $\mathrm{SWEP}_{n,m}$ with loss of regularity ✓ [Bierstone–Campesato–Milman 2021]; in o-minimal expansions of the real field by restricted quasianalytic functions.

O-minimal expansions of the real field

This is a family $\mathscr{S} = (\mathscr{S}_n)_{n\geq 1}$, where $\mathscr{S}_n \subseteq \mathscr{P}(\mathbb{R}^n)$ such that

- $\mathscr{S}_n$ is a boolean algebra with respect to the usual set-theoretic operations,
- $\mathscr{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$,
- $\mathscr{S}$ is stable by cartesian products and linear projections,
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Model-theoretic definition

A structure $(\mathbb{R}, -, +, \cdot, <, 0, 1, \ldots)$ is o-minimal if every $X \subseteq \mathbb{R}$ given by a first-order formula of the structure is a finite union of intervals and points.

Examples

- Semialgebraic sets: $\mathbb{R}_{\mathrm{sa}} = (\mathbb{R}, -, +, \cdot, <, 0, 1)$ [Tarski 1930]
- Globally subanalytic sets: $\mathbb{R}_{\mathrm{an}} := (\mathbb{R}_{\mathrm{sa}}, \text{restricted analytic functions})$ [Gabrielov 1968]
- $\mathbb{R}_{\exp} := (\mathbb{R}_{\mathrm{sa}}, \exp)$ [Wilkie 1991]
- $\mathbb{R}_{\mathrm{an},\exp} := (\mathbb{R}_{\mathrm{an}}, \exp)$ [van den Dries–Miller 1994]

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Some properties

- Finitely many connected components which again are definable.
- Monotonicity theorem, cell decomposition theorem, etc.
- Stability under composition, implicit and inverse functions.
- Derivatives of definable functions are definable, but not antiderivatives.
- Miller's dichotomy.

Table of Contents

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Remarks

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- For $m = 1$ this is due to [Brudnyi–Shvartsman 2001] with the optimal $k = 3 \cdot 2^{n-1}$.
- A variant of this result is crucial for Fefferman's solution of $\mathrm{WEP}_{n,m}$.

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- Function spaces:

$$\begin{aligned} C^{m,\omega}(\mathbb{R}^n) &:= \{f \in C^m(\mathbb{R}^n) : |f^{(\alpha)}(x)| \lesssim 1, \text{ for } |\alpha| \le m,\ x \in \mathbb{R}^n, \\ &\qquad |f^{(\alpha)}(x) - f^{(\alpha)}(y)| \lesssim \omega(|x-y|), \text{ for } |\alpha| = m,\ x, y \in \mathbb{R}^n\} \\ C^{m,\omega}_{\mathrm{def}}(\mathbb{R}^n) &:= \{f \in C^{m,\omega}(\mathbb{R}^n) : f \text{ definable}\} \\ C^{m,\omega}(\mathbb{R}^n)|_X &:= \{f : X \to \mathbb{R} :\ \exists F \in C^{m,\omega}(\mathbb{R}^n),\ F|_X = f\} \\ C^{m,\omega}_{\mathrm{def}}(\mathbb{R}^n)|_X &:= \{f : X \to \mathbb{R} :\ \exists F \in C^{m,\omega}_{\mathrm{def}}(\mathbb{R}^n),\ F|_X = f\} \\ \mathbb{R}^X_{\mathrm{def}} &:= \{f : X \to \mathbb{R} : f \text{ definable}\} \end{aligned}$$

All spaces are equipped with their natural norms.

Theorem A [Parusiński–R 2023]

Let ω be a definable modulus of continuity (e.g. any t^α with $\alpha \in (0,1] \cap \mathbb{Q}$), $f : X \to \mathbb{R}$ definable, $X \subseteq \mathbb{R}^n$ closed. TFAE:

1. f extends to a definable $C^{1,\omega}$ function on $\mathbb{R}^n$.
2. f extends to a $C^{1,\omega}$ function on $\mathbb{R}^n$.
3. For all $Y \subseteq X$, $\#Y \leq 3 \cdot 2^{n-1}$ there is $F_Y \in C^{1,\omega}(\mathbb{R}^n)$ such that $F_Y|_Y = f|_Y$ and $\sup_Y \|F_Y\|_{C^{1,\omega}(\mathbb{R}^n)} < \infty$. [Brudnyi–Shvartsman 2001]

That means

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In the Lipschitz case $\omega(t) = t$, for compact definable X:

$$\|f\|_{C^{1,1}_{\mathrm{def}}(\mathbb{R}^n)|_X} \approx_n \|f\|_{C^{1,1}(\mathbb{R}^n)|_X}.$$

Theorem B [Parusiński–R 2023]

- Let $(\mathcal{M}, \rho)$ be a definable pseudometric space, i.e. $\mathcal{M} \subseteq \mathbb{R}^N$ and ρ are definable, and $\mathrm{Aff}_k(\mathbb{R}^n) := \{\text{affine } H \subseteq \mathbb{R}^n : \dim H \leq k\}$.
- Let $F : \mathcal{M} \to \mathrm{Aff}_k(\mathbb{R}^n)$ be a definable map, i.e. its graph defined as $\Gamma(F) = \bigcup_{x \in X} (\{x\} \times F(x))$ is definable.

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- Let $(\mathcal{M}, \rho)$ be a definable pseudometric space, i.e. $\mathcal{M} \subseteq \mathbb{R}^N$ and ρ are definable, and $\mathrm{Aff}_k(\mathbb{R}^n) := \{\text{affine } H \subseteq \mathbb{R}^n : \dim H \leq k\}$.
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The following assertions are equivalent:

1. F has a definable Lipschitz selection $f : \mathcal{M} \to \mathbb{R}^n$ (i.e. $\Gamma(f) \subseteq \Gamma(F)$).
2. F has a Lipschitz selection $\hat{f} : \mathcal{M} \to \mathbb{R}^n$.
3. For all $\mathcal{N} \subseteq \mathcal{M}$, $\#\mathcal{N} \leq 2^{k+1}$, there is a Lip-selection $f_{\mathcal{N}}$ of $F|_{\mathcal{N}}$ such that $\sup_{\mathcal{N}} |f_{\mathcal{N}}|_{\mathrm{Lip}(\mathcal{N}, \mathbb{R}^n)} < \infty$. [Brudnyi–Shvartsman 2001]

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If $\hat{f}$ is a Lip-selection of F, then there is a definable Lip-selection f of F with

$$|f|_{\mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)} \leq C(k, n)\, |\hat{f}|_{\mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)}.$$

Theorem C [Parusiński–R 2023]

Let ω be a definable modulus of continuity and $A_{ij}, b_i : X \to \mathbb{R}$, $1 \le i \le N$, $1 \le j \le M$, definable functions on $X \subseteq \mathbb{R}^n$. Consider

$$\sum_{j=1}^{M} A_{ij} f_j = b_i, \quad 1 \le i \le N. \qquad (\dagger)$$

The following assertions are equivalent:

1. ($\dagger$) has a definable ω-Hölder solution.
2. ($\dagger$) has an ω-Hölder solution.

If $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_M)$ is an ω-Hölder solution, then there is a definable ω-Hölder solution $f = (f_1, \ldots, f_M)$ with

$$|f|_{C^{0,\omega}(X,\mathbb{R}^M)} \le C(M)\, |\hat{f}|_{C^{0,\omega}(X,\mathbb{R}^M)}.$$

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Remark

A C^0-version is due to [Aschenbrenner–Thamrongthanyalak 2019], in the semialgebraic setting see also [Fefferman–Kollár 2013].

Theorem A:
$C^{1,\omega}$ extension

Theorem B:
Lipschitz selection

Theorem C:
BEHK problem

Theorem A: $C^{1,\omega}$ extension

Theorem B: Lipschitz selection → **Theorem C**: BEHK problem

Theorem B implies Theorem C

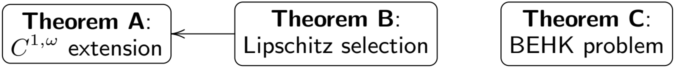
- For $x \in X$, set

$$F(x) := \{(f_1, \ldots, f_M) \in \mathbb{R}^M : \sum_{j=1}^{M} A_{ij}(x) f_j = b_i(x),\ 1 \le i \le N\}.$$

- Then $F : X \to \mathrm{Aff}_M(\mathbb{R}^M)$ is definable.
- Equip X with the metric $\rho(x,y) := \omega(\|x-y\|)$.
- That (†) has a (definable) ω-Hölder solution means precisely that F has a (definable) Lipschitz selection.
- Apply Theorem B. □

Reducing Theorem A to Theorem B

(following [Brudnyi–Shvartsman 2001])



To show

- If $f \in \mathbb{R}^X_{\text{def}} \cap C^{1,\omega}(\mathbb{R}^n)|_X$ then $f \in C^{1,\omega}_{\text{def}}(\mathbb{R}^n)|_X$.
- Boundedness of $\mathbb{R}^X_{\text{def}} \cap C^{1,\omega}(\mathbb{R}^n)|_X = C^{1,\omega}_{\text{def}}(\mathbb{R}^n)|_X$.
- In the Lipschitz case, for compact X, $\|f\|_{C^{1,1}_{\text{def}}(\mathbb{R}^n)|_X} \approx_n \|f\|_{C^{1,1}(\mathbb{R}^n)|_X}$.

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Strategy

Associate a definable affine-set valued map in such a way that it admits a definable Lipschitz selection if and only if f can be completed to a definable Whitney jet (f, g) of class $C^{1,\omega}$.

Whitney jets of class $C^{1,\omega}$ on X

These are pairs (f, g), where $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^n$ satisfy

$$\|(f,g)\|_{X,1,\omega} := \sup_{x \in X} |f(x)| + \sup_{x \in X} \|g(x)\| + |(f,g)|_{X,1,\omega} < \infty$$

$$|(f,g)|_{X,1,\omega} := \sup_{x \neq y \in X} \frac{|f(x) - f(y) - \langle g(y), x - y \rangle|}{\|x - y\| \, \omega(\|x - y\|)} + \sup_{x \neq y \in X} \frac{\|g(x) - g(y)\|}{\omega(\|x - y\|)}$$

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Associated affine-set valued map

Given $X \subseteq \mathbb{R}^n$ closed definable, consider $(\mathcal{M}_X, \rho_\omega)$ where

$$\mathcal{M}_X := \{(x,y) \in X \times X : x \neq y\},$$

$$\rho_\omega((x,y),(x',y')) := \omega(\|x-y\|) + \omega(\|x'-y'\|) + \omega(\|x-x'\|) \text{ if } (x,y) \neq (x',y'),$$

Given $f : X \to \mathbb{R}$ definable bounded, consider $L_f : \mathcal{M}_X \to \mathrm{Aff}_{n-1}(\mathbb{R}^n)$ with

$$L_f(x,y) := \{z \in \mathbb{R}^n : f(x) = f(y) + \langle z, x-y\rangle\}.$$

Proposition

Assume $\omega \leq 1$. The following assertions are equivalent:

1. There is a bounded definable $g : X \to \mathbb{R}^n$ such that (f, g) is a definable Whitney jet of class $C^{1,\omega}$ on X.
2. There is a <u>bounded</u> definable Lip-selection ℓ of L_f.

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3. There is a definable Lip-selection $\widetilde{\ell}$ of $\widetilde{L}_f : \widetilde{\mathcal{M}}_X \to \mathcal{A}_{n-1}(\mathbb{R}^n)$, where $\widetilde{\mathcal{M}}_X = \mathcal{M}_X \cup \{*\}$, $\widetilde{\rho}_\omega(m, *) := 2$, and $\widetilde{L}_f(*) := \{0\}$.

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If these equivalent conditions hold, then

$$\inf_g \|(f,g)\|_{X,1,\omega} \approx_n \sup_{x \in X} |f(x)| + \inf_\ell \left\{ \sup_{(x,y) \in \mathcal{M}_X} \|\ell(x,y)\| + |\ell|_{\mathrm{Lip}(\mathcal{M}_X, \mathbb{R}^n)} \right\}$$
$$\approx_n \sup_{x \in X} |f(x)| + \inf_{\widetilde{\ell}} |\widetilde{\ell}|_{\mathrm{Lip}(\widetilde{\mathcal{M}}_X, \mathbb{R}^n)}.$$

Sketch of the proof

- Let $f \in \mathbb{R}^X_{\mathrm{def}} \cap C^{1,\omega}(\mathbb{R}^n)|_X$.

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- This follows from the assumption, since the above proposition also holds if the attribute "definable" is removed.
- Definable Whitney jets of class $C^{1,\omega}$ extend to definable $C^{1,\omega}$ functions in a bounded way.

Theorem D [Parusiński–R 2023]

Let $0 \leq m \leq p$ be integers, ω a modulus of continuity, $X \subseteq \mathbb{R}^n$ closed definable. Any definable bounded family of $C^{m,\omega}$ Whitney jets on X extends to a definable bounded family of $C^{m,\omega}$ functions on $\mathbb{R}^n$, C^p outside X.

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- **Corollary**: C^m version with boundedness for definable families $(X_a)_{a\in A}$ of compact $X_a \subseteq \mathbb{R}^n$.

Open problem

Is there a continuous and/or linear extension operator? [Pawłucki 2008]

Gromov's inequality

Let $\varphi : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, be definable. There exists a definable closed subset $Z \subseteq U$ with $\dim Z < n$ such that φ is C^p on $U \setminus Z$ and for each ball $B = B(x, r)$ in $U \setminus Z$

$$|\partial^\alpha \varphi(x)| \leq C(n, p) \sup_{y \in B} |\varphi(y)| \, r^{-|\alpha|}, \quad |\alpha| \leq p.$$

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Uniform Λ_p stratification

Definable families of sets admit a stratification into a finite number of cells that are defined by functions satisfying bounds of the above type. The appearing constants and the number of cells are independent of the parameter of the family.

To show

Let $X \subseteq \mathbb{R}^n$ be definable and compact. Then

$$\|f\|_{C^{1,1}_{\mathrm{def}}(\mathbb{R}^n)|_X} \approx_n \|f\|_{C^{1,1}(\mathbb{R}^n)|_X}.$$

To show

Let $X \subseteq \mathbb{R}^n$ be definable and compact. Then

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To this end, one needs to control the $C^{1,1}$-norm of the definable extension F of a definable $C^{1,1}$-Whitney jet (f,g):

$$\|F\|_{C^{1,1}(\mathbb{R}^n)} \le C(n)\,\|(f,g)\|_{X,1,1}.$$

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Theorem [Azagra–Le Gruyer–Mudarra 2018]

Given a $C^{1,1}$ Whitney jet (f, g) on $X \subseteq \mathbb{R}^n$ with $|(f,g)|_{X,1,1} \leq M$, then a $C^{1,1}$ extension F of (f, g) to $\mathbb{R}^n$ can be given by an explicit formula such that $F|_X = f$, $\nabla F|_X = g$, and $|\nabla F|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)} \leq M$.

Formula for F

$$F(x) := \operatorname{conv}(h)(x) - \frac{M}{2}\|x\|^2, \quad x \in \mathbb{R}^n$$

$$h(x) := \inf_{y \in X} \left\{ f(y) + \langle g(y), x - y \rangle + \frac{M}{2}\|x - y\|^2 \right\} + \frac{M}{2}\|x\|^2$$

$$\operatorname{conv}(h) := (h^*)^* \quad \text{the convex envelop of } h$$

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If X, f, and g are definable, then F is definable.

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Remark

Based on the fact: $f : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,1}$ with $|\nabla f|_{\mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)} = M$ if and only if $f + \frac{M}{2}\|\cdot\|^2$ is convex and $f - \frac{M}{2}\|\cdot\|^2$ is concave.

Theorem [Azagra–Le Gruyer–Mudarra 2020]

Let $f : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$, be a Lipschitz function with $M = |f|_{\mathrm{Lip}(X,\mathbb{R}^m)}$.
Then there is a Lipschitz extension $F : \mathbb{R}^n \to \mathbb{R}^m$ of f with $|F|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^m)} = M$ given by an explicit formula:

$$F(x) := \nabla_{\mathbb{R}^m} \operatorname{conv}(g)(x,0), \quad x \in \mathbb{R}^n$$

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$$(x,y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

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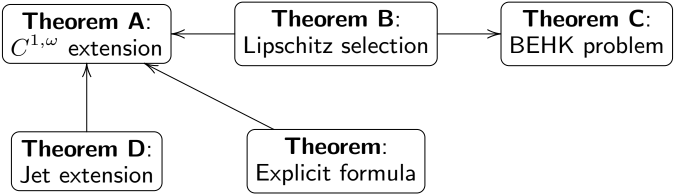
Corollary (Definable Kirszbraun theorem) [Aschenbrenner–Fischer 2010]

If f is definable, then F is a definable Lipschitz extension of f with $|F|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^m)} = |f|_{\mathrm{Lip}(X,\mathbb{R}^m)}$.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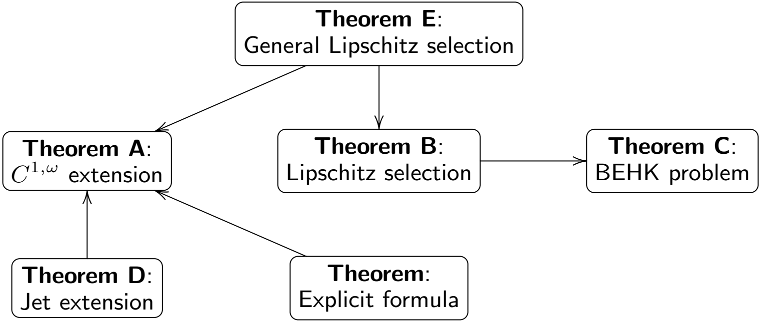
Theorem E:
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- Endow (V, E) with a weight, i.e., a symmetric function $w : E \to [0, \infty]$. It induces an extended pseudometric space (V, σ). If $v \neq v'$, then $\sigma(v, v')$ is the infimum of the sums of weights over all paths of edges joining v and v'.

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- We say that (V, E, w) is a definable weighted graph if there is a definable pseudometric $\rho : V \times V \to [0, \infty)$ and $A \geq 1$ such that

$$\rho/A \leq \sigma \leq A\rho.$$

  Then (V, ρ) is a definable pseudometric space. Any two vertices are connected by a path of edges and V is an admissible subset of V.

Theorem E [Parusiński–R 2023]

Let (V, E, w) be a definable weighted graph, $F : V \to \mathrm{Aff}_k(\mathbb{R}^n)$ definable.
Assume that for each admissible $W \subseteq V$ with $\#W \leq 2^{k+1}$ there is a Lip-selection $f_W : W \to \mathbb{R}^n$ of $F|_W$ such that $|f_W|_{\mathrm{Lip}(W,\mathbb{R}^n)} \leq 1$.
Then there exists a definable Lip-selection $f : V \to \mathbb{R}^n$ of F such that

$$|f|_{\mathrm{Lip}(V,\mathbb{R}^n)} \leq C(k, n, A).$$

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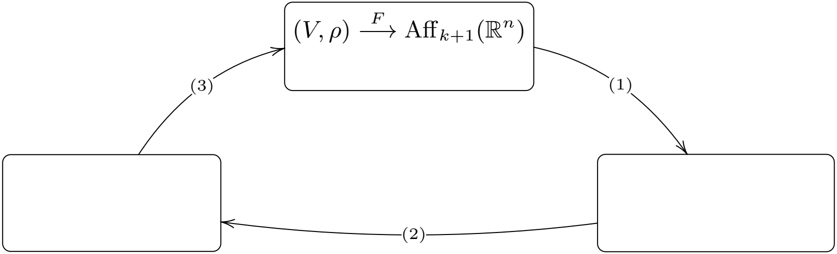
Theorem E implies Theorem B

If $(\mathcal{M}, \rho)$ is a definable pseudometric space, then the full graph $(\mathcal{M}, \{(m, m') \in \mathcal{M} \times \mathcal{M} : m \ne m'\})$ with weight ρ is a definable weighted graph (where $\rho = \sigma$). Any subset of $\mathcal{M}$ is admissible. $\square$

Based on [Brudnyi–Shvartsman 2001]. Induction on k. The induction step ($k \to k+1$):



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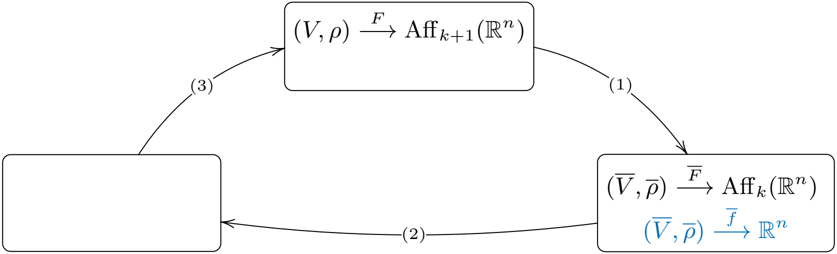

1. From F one constructs a "doubling" $(\overline{V},\overline{\rho})$ of (V,ρ) and $\overline{F}$ such that there is a definable Lip-selection $\overline{f}$ of $\overline{F}$, by induction.

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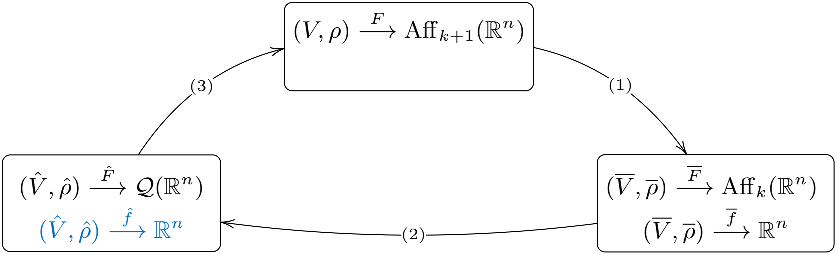

1. From F one constructs a "doubling" $(\overline{V}, \overline{\rho})$ of (V, ρ) and $\overline{F}$ such that there is a definable Lip-selection $\overline{f}$ of $\overline{F}$, by induction.
2. $\overline{f}$ is used to define a new space $(\hat{V} = \overline{V}, \hat{\rho})$ and a cube-valued map $\hat{F} : \hat{V} \to \mathcal{Q}(\mathbb{R}^n)$; the center of the cube $\hat{F}(\overline{v})$ is $\overline{f}(\overline{v})$. There is a Lip-selection $\hat{f}$ of $\hat{F}$; it can be interpreted as a Lipschitz map on (V, ρ).

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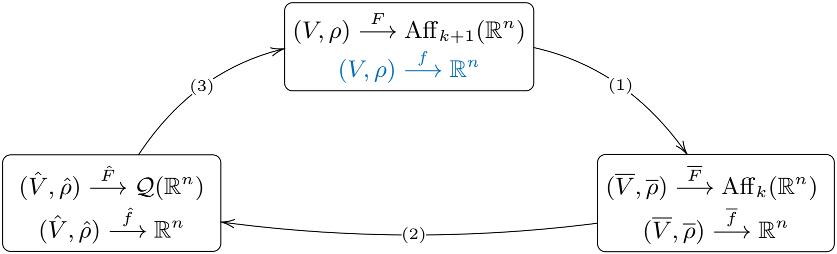

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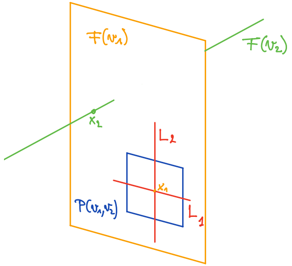
3. The desired Lip-selection f of F is found by defining $f(v)$ to be the orthogonal projection of $\hat{f}(v)$ to the affine subspace $F(v)$ of $\mathbb{R}^n$.

Suppose that $v_1 \leftrightarrow v_2$ are joined by an edge. By assumption, there exist $x_i \in F(v_i)$ such that $\|x_1 - x_2\| \leq \rho(v_1, v_2)$.

$$\begin{aligned} P(v_1, v_2) &= F(v_1) \cap (F(v_2) + Q(2\rho(v_1, v_2)) + x_1 - x_2) \\ &= \bigcap_{i \in I(v_1, v_2)} F(v_1) \cap (L_i + Q(r_i)) \end{aligned}$$

$L_i \subseteq F(v_1)$ are affine subspaces of dimension $\leq k$ containing x_1.



$\overline{V} := \{(v_1, v_2, i) : v_1 \leftrightarrow v_2, i \in I(v_1, v_2)\}$

$\overline{\rho}(\overline{v}, \overline{v}') := \rho(v_1, v_1') + r_i + r_i'$

$\overline{F} : \overline{V} \to \mathrm{Aff}_k(\mathbb{R}^n)$, $\overline{F}(v_1, v_2, i) := L_i$

Open problems

- $\mathrm{SWEP}_{n,m}$ for $m \geq 2$, $n \geq 3$
- Control of the (semi)norms
- Control of the semialgebraic diagram
- Extension of semialgebraic/definable $C^{m,\omega}$ Whitney jets by a continuous and/or linear operator
- ...

Happy Birthday!