On Gabrielov's rank Theorem

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June 25th, 2025

Images of algebraic maps

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- In particular the dimension of $F(X)$ equals the dimension of its Zariski closure $\overline{F(X)}$:

$$\dim(\overline{F(X)}) = \dim(F(X)) \leq \dim(X).$$

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  Then the Zariski closure of $\Phi(\mathbb{C}^2)$ equals $\mathbb{C}^3$.

  Therefore, $\Phi(\mathbb{C}^2)$ is not analytic.

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  We always have $\boxed{\mathrm{r}(\Phi) \leq \mathrm{r}^{\mathcal{F}}(\Phi) \leq \mathrm{r}^{\mathcal{A}}(\Phi)}$

Analytic vs formal category: Gabrielov's Example (1973)

Answering a question of Grothendieck, Gabrielov provides an example of a morphism

$$\psi : \mathbb{C}\{x_1, x_2, x_3, x_4\} \longrightarrow \mathbb{C}\{u, v\}$$

with

$$\mathrm{r}(\psi) = 2 < \mathrm{r}^{\mathcal{F}}(\psi) = 3 < \mathrm{r}^{\mathcal{A}}(\psi) = 4$$

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- **Question:** Can we extend this result to a more general (and algebraic) setting?

Weierstrass temperate families

We fix an uncountable algebraically closed field of characteristic zero $\mathcal{K}$.
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- (Weierstrass division) $\mathcal{K}\{\{x_1, \ldots x_n\}\}$ satisfies the Weierstrass division Theorem.
- (Stability under blow-down) For $f \in \mathcal{K}[[x_1, \ldots, x_n]]$

$$f(x_1, \ldots, x_{n-1}, x_1 x_n) \in \mathcal{K}\{\{x\}\} \Longrightarrow f \in \mathcal{K}\{\{x\}\}.$$

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- (Stability under hyperplane sections) For $f \in \mathcal{K}[\![x]\!] \setminus \mathcal{K}\{\{x\}\}$, the set

$$\{\lambda \in \mathcal{K} \mid f(x_1, \ldots, x_{n-1}, \lambda x_1) \in \mathcal{K}\{\{x\}\}\}$$

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- ("Temperateness"): Let $\gamma(t)$ be algebraic of degree d over $\mathcal{K}[t]$. Let $a_1(t, z), \dots, a_d(t, z) \in \mathcal{K}[t][\![z]\!]$. Then

$$a_d(t, z) + a_{d-1}(t, z)\gamma(t) + \cdots + a_1(t, z)\gamma(t)^{d-1} \in \mathcal{K}\{\{t, z\}\}$$

$$\Longrightarrow a_i(t, z) \in \mathcal{K}\{\{t, z\}\}.$$

Rank Theorem (Belotto, Curmi, R., 2022):

Let

$$\varphi : \mathcal{K}\{\{x_1, \ldots, x_n\}\} \longrightarrow \mathcal{K}\{\{u_1, \ldots, u_p\}\}$$

be a morphism of temperate power series rings. Then

$$\mathrm{r}(\Phi) = \mathrm{r}^{\mathcal{F}}(\Phi) \Longrightarrow \mathrm{r}^{\mathcal{F}}(\Phi) = \mathrm{r}^{\mathcal{T}}(\Phi).$$

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(iv) (Eisenstein series) Let A be a UFD containing an uncountable characteristic zero field.

$$\mathcal{K} = \text{ fraction field of } A$$

Then

$$\mathcal{K}\{\{x_1, \ldots, x_n\}\} := \bigcup_{f\in A\setminus\{0\}} A_f[\![x_1, \ldots, x_n]\!]$$

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Strategy of proof (1)

We may reduce to the case $\psi : \mathcal{K}\{\{x_1, x_2, y\}\} \longrightarrow \mathcal{K}\{\{u, v\}\}$ is such that

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Moreover we may assume that

$$\psi : (x_1, x_2, y) \longmapsto (u, uv, f(u, v))$$

for some $f(u, v) \in \mathcal{K}\{\{u, v\}\}$.

A bit of algebra: roots of polynomials with power series coefficients

We denote by $\mathbb{P}[\![x]\!]$ the set of series of the form

$$\sum_{k=0}^{\infty} \frac{a_k(x)}{b(x)^{\alpha k+\beta}}$$

where the a_k and b are homogeneous polynomials, the total degree of $\frac{a_k(x)}{b(x)^{\alpha k+\beta}}$ is k, and α, $\beta \in \mathbb{N}$.

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- **Theorem** (Tougeron 90): If $P(x, y) \in \mathcal{K}[\![x]\!][y]$ is a monic polynomial in y, then its roots can be expressed as

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- Then $\sigma_{\mathfrak{a}}^*(A)$ is a power series.
- Moreover $\sigma_{\mathfrak{a}}^*(A) \in \mathcal{K}\{\{u, v\}\}$ iff $A \in \mathbb{P}\{\{x\}\}$.

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We have

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By assumption, $P\left(x_1, x_2, f\left(x_1, \dfrac{x_2}{x_1}\right)\right) = 0$

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We make more blowups in order to insure that $\Delta_y P_{\mathfrak{b}}$ is normal crossing at every point $\mathfrak{b}$ of the exceptional divisor.

$$(N_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (N_1, E_1) \xrightarrow{\sigma_1} (\mathcal{K}^2, 0)$$

We set $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$ and, $\forall j \in \{1, \dots, r\}$, E_j is a simple normal crossing divisor:

$$E_j = E_j^{(1)} \cup E_j^{(2)} \cup \cdots \cup E_j^{(j)}$$

where $E_j^{(k)}$ is the strict transform of $E_{j-1}^{(k)}$ ($k < j$) and $E_j^{(j)}$ is the exceptional divisor of σ_j.

Strategy of proof (4)

Claim 1: let Q be one of the Q_i. Then, for every $\mathfrak{a} \in E_r^{(1)}$, $\sigma_{\mathfrak{a}}^*(Q) \in \mathcal{K}[[x_1', x_2']][y]$ for some local coordinates (x_1', x_2') at $\mathfrak{a}$.
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If $r = 1$: ok

Proof by induction on (r, k):

$$r = \text{number of blowups}$$

$$P \text{ has a temperate factor at } \mathfrak{a} \in E_r^{(k)}$$