Tame geometry and extension of functions

Pawłucki 70

The Nash-Tognoli theorem over $\mathbb{Q}$

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$\mathbb{Q}$-algebraicity problem: (Parusiński, '21)

Is every algebraic set $X \subset \mathbb{R}^n$ homeomorphic to some algebraic set $X' \subset \mathbb{R}^m$ defined by polynomial equations with rational coefficients?

estimates depending on $\dim(X)$

additional regularity + approximation

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§ Description over $\overline{\mathbb{Q}}^r$

Denote by $\overline{\mathbb{Q}}^r := \overline{\mathbb{Q}} \cap \mathbb{R}$ the field of real algebraic numbers.

Theorem: (Parusiński-Rond, '20) Let $X \subset \mathbb{R}^n$ be an algebraic set.

Then, there exists an algebraic set $X' \subset \mathbb{R}^n$ such that:

1. There is an homeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ such that $h(X) = X'$.

2. h is semialgebraic and arc-analytic.

3. If $p_1, \ldots, p_\ell \in \mathbb{R}[x]$ are such that $X = \mathcal{Z}_{\mathbb{R}}(p_1, \ldots p_\ell)$, then $X' = \mathcal{Z}_{\mathbb{R}}(q_1, \ldots, q_\ell)$ with $q_1, \ldots q_\ell \in \overline{\mathbb{Q}}^r[x]$ and each q_i approximates p_i.

The proof uses a Zariski-equisingular deformation of the coefficients of the polynomials $p_1(x), \dots, p_\ell(x) \in \mathbb{R}[x]$ s.t. $Z_{\mathbb{R}^n}(p_1, \dots, p_\ell) = X$.

CRUCIAL PROPERTY: (Model completeness of the theory of real closed fields)

Let $S = \bigcup_{i=1}^{t} \bigcap_{j=1}^{s_i} \{ q_{ij}(x) *_{ij} 0 \} \subset \mathbb{R}^n$ with $*_{ij} \in \{=, <\}$, $q_{ij} \in \overline{\mathbb{Q}}^r[x]$

be a semialgebraic set then:

$$S \cap (\overline{\mathbb{Q}}^r)^n \neq \emptyset \iff S \neq \emptyset.$$

Counterexample (Teissier '90, Parusiński–Păunescu '25)

There is a surface singularity $(X,0) \subset (\mathbb{C}^3,0)$ whose equisingular class does not admit any $\mathbb{Q}$-algebraic representative.

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Easy case: Compact nonsingular algebraic hypersurfaces

$M \subset \mathbb{R}^n$ cpt nonsingular algebraic hypersurface, choose:

(i) $p \in \mathbb{R}[x]$ s.t. $\mathcal{Z}_{\mathbb{R}^n}(p) = M$ & $\nabla p(a) \neq 0 \ \forall a \in M$

(i)' $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ s.t. $f|_{U'} = p|_{U'}$ & $f|_{\mathbb{R}^n \setminus U} \equiv 1$.

(ii) $U' \subset U \subset \mathbb{R}^n$ a cpt neighborhood of M in $\mathbb{R}^n$.

Uniform
Weierstrass approximation + Thom isotopy lemma

$\Longrightarrow$ there is $g \in \mathbb{Q}[x]$ s.t. $M' = \mathcal{Z}_{\mathbb{R}^n}(g) \cap U \cong M$.
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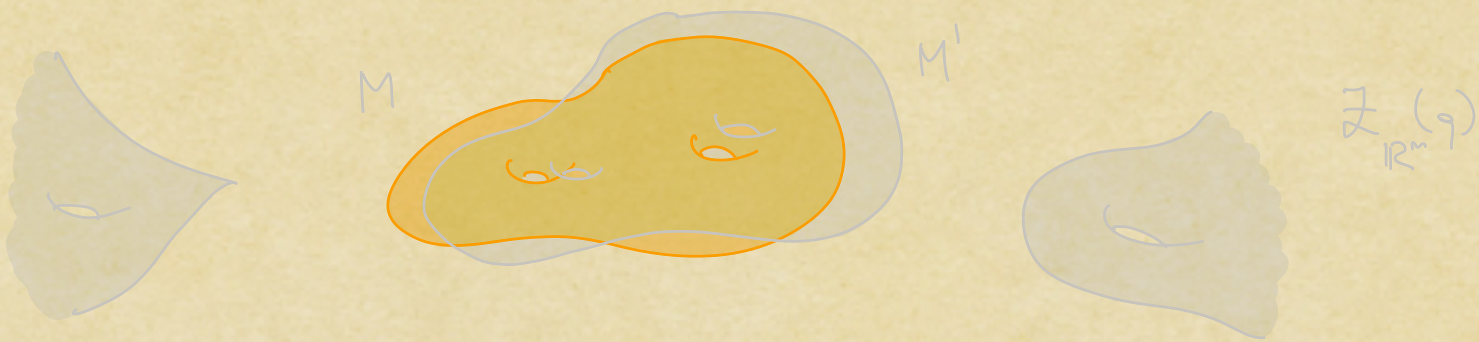

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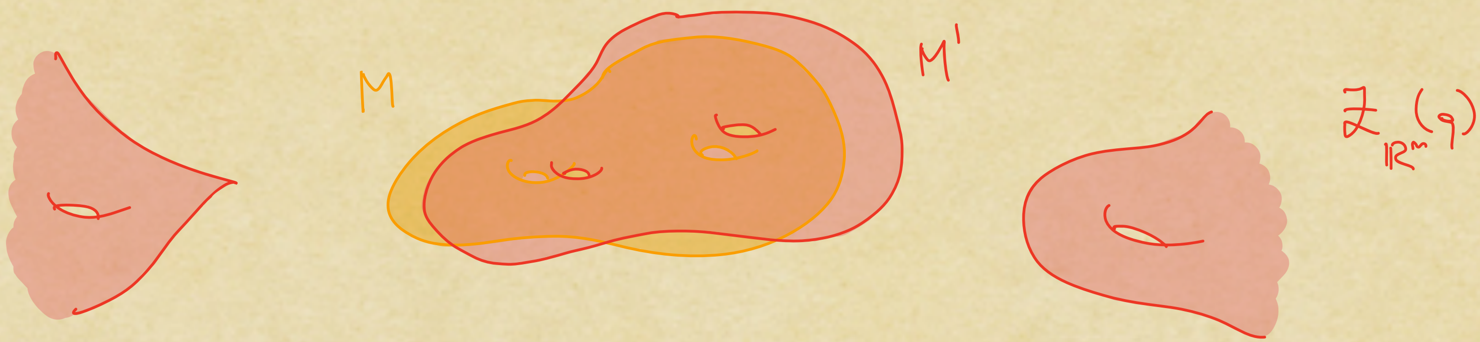

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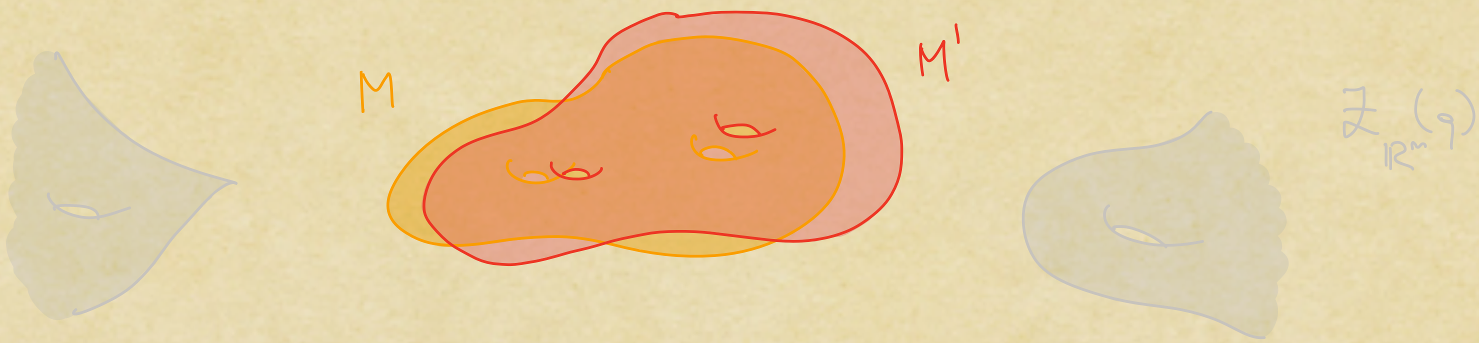

§ $\mathbb{Q}$-regularity

Let $X \subset \mathbb{R}^n$ be a $\mathbb{Q}$-algebraic set and $a = (a_1, \ldots, a_n) \in X$. Let:

$$\mathfrak{n}_a := (x_1 - a_1, \ldots, x_n - a_n) \quad \text{and} \quad \mathcal{I}_{\mathbb{Q}}(X) := \mathcal{I}(X) \cap \mathbb{Q}[x].$$

Definition: (Fernando-Ghiloni) We define the $\mathbb{Q}$-local ring

$$\mathcal{R}^{\mathbb{Q}}_{X,a} := \mathbb{R}[x]_{\mathfrak{n}_a} / \mathcal{I}_{\mathbb{Q}}(X)\,\mathbb{R}[x]_{\mathfrak{n}_a}.$$

We say that $a \in \mathrm{Reg}(X)$ is $\mathbb{Q}$-regular if the ring $\mathcal{R}^{\mathbb{Q}}_{X,a}$ is a regular local ring of dimension $\dim(X)$. We denote by $\mathrm{Reg}^{\mathbb{Q}}(X) \subset \mathrm{Reg}(X)$ the set of $\mathbb{Q}$-regular points of X.

Remark: $\emptyset \neq \mathrm{Reg}^{\mathbb{Q}}(X) \subset \mathrm{Reg}(X)$ is Zariski open & $\mathrm{Sing}^{\mathbb{Q}}(X) = X \setminus \mathrm{Reg}^{\mathbb{Q}}(X)$ is $\mathbb{Q}$-algebraic.

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Definition: Let $X \subset \mathbb{R}^n$ be a $\mathbb{Q}$-algebraic set. We say that $X \subset \mathbb{R}^n$ is

- **$\mathbb{Q}$-determined** if $\mathrm{Reg}(X) = \mathrm{Reg}^{\mathbb{Q}}(X)$.
- $\mathbb{Q}$-nonsingular if $X = \mathrm{Reg}^{\mathbb{Q}}(X)$.

$\mathbb{Q}$-Jacobian criterion: Let X be a $\mathbb{Q}$-algebraic set. Then,

$a \in \mathrm{Reg}^{\mathbb{Q}}(X) \iff$ there are $q_1, \dots, q_{n-d} \in \mathcal{I}_{\mathbb{Q}}(X)$ such that

(i) $\nabla q_1(a), \dots, \nabla q_{n-d}(a)$ are linearly independent over $\mathbb{R}$

(ii) there is an Euclidean neighborhood U of a in $\mathbb{R}^n$ such that

$$X \cap U = \mathcal{Z}_{\mathbb{R}}(q_1, \dots, q_{n-d}) \cap U.$$

Definition: Let $X \subset \mathbb{R}^m$ be a $\mathbb{Q}$-algebraic set. We say that $X \subset \mathbb{R}^m$ is

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Lemma (Weierstrass approximation of $\mathcal{C}^\infty$-functions)

Let $X \subset \mathbb{R}^n$ be a compact $\mathbb{R}$-nonsingular $\mathbb{Q}$-algebraic set. Let $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $X \subset \mathcal{Z}_{\mathbb{R}^n}(f)$. Then, there is $q \in \mathcal{I}_{\mathbb{Q}}(X)$ arbitrarily $\mathcal{C}^\infty_w$-close to f.

proof: Let $\mathcal{I}_{\mathbb{Q}}(X) = (q_1, \dots, q_\ell)$ and $a \in \mathbb{R}^n$. Then, there is a neighborhood of a in $\mathbb{R}^n$ and $u_1, \dots, u_\ell \in \mathcal{C}^\infty(U_a)$ such that

$$f|_{U_a} = \sum_{i=1}^{\ell} u_i \cdot q_i .$$

(1) If $a \in X$, by the $\mathbb{R}$-Jacobian criterion, there are

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Lemma (Weierstrass approximation of $\mathcal{C}^\infty$-functions)

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(ii) $\mathcal{U}_a \cap X = \mathcal{U}_a \cap \mathcal{Z}_{\mathbb{R}^n}(q_{i_1}, \ldots, q_{i_{n-d}})$.

$\Longrightarrow$ there are a neighborhood $\mathcal{U}_a$ and $u_{i_1}, \ldots, u_{i_{n-d}} \in \mathcal{C}^\infty(\mathcal{U}_a)$ s.t.

$$f = \sum_{j=1}^{n-d} u_{i_j} \, q_{i_j} .$$

Then, just fix $u_i = 0$ for every $i \notin \{i_1, \ldots, i_{n-d}\}$.

(2) If $a \notin X$, then there is $q_i \in \{q_1, \ldots, q_\ell\}$ s.t. $q_i(a) \neq 0$

$\Longrightarrow$ $f = \frac{f}{q_i} \cdot q_i$, so take $u_i = \frac{f}{q_i}$, $u_j = 0$ $\forall j \neq i$.

and $\mathcal{U}_a = \mathbb{R}^n \setminus \mathcal{Z}_{\mathbb{R}^n}(q_i)$.

Patch together the local descriptions by a partition of unity argument getting

$f = \sum_{i=1}^{\ell} u_i' \cdot q_i$, then approximate each $u_i \in \mathcal{C}^\infty(\mathbb{R}^n)$ by $p_i \in \mathbb{Q}[x]$ over K.

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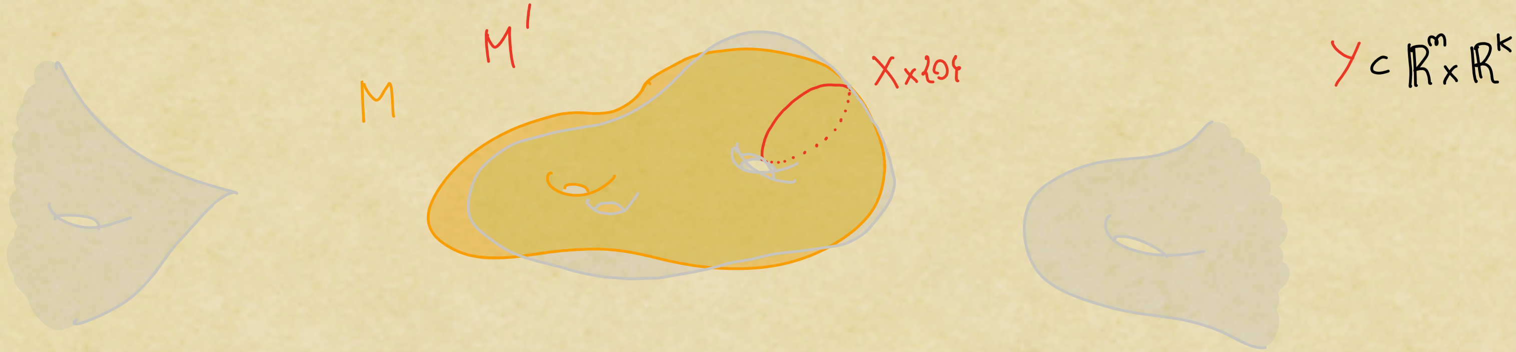
Nash theorem over $\mathbb{Q}$: (Ghiloni, S.) Let $M \subset \mathbb{R}^n$ be a compact $\mathcal{C}^\infty$ manifold and let $X \subset M$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.

Then, M is diffeomorphic fixing X to the union of some connected components M' of a $\mathbb{Q}$-algebraic set $Y \subset \mathbb{R}^n \times \mathbb{R}^k$ such that:

$$X \times \{0\} \subset M' \qquad \& \qquad M' \subset \mathrm{Reg}^{\mathbb{Q}}(Y).$$



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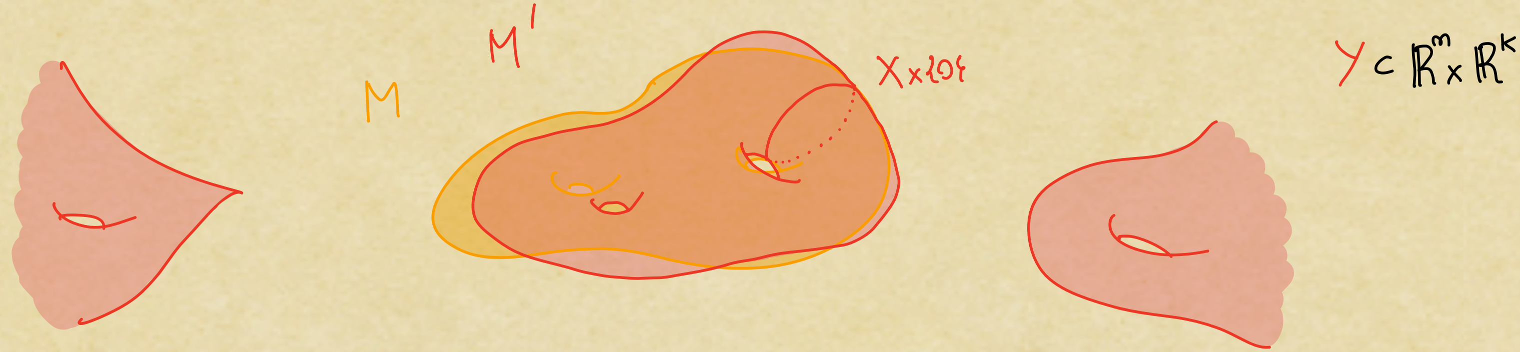

Nash-Tognoli theorem over $\mathbb{Q}$: (Ghiloni-S.)

Let $M \subset \mathbb{R}^n$ be a compact $\mathscr{C}^\infty$-manifold. There exists a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset \mathbb{R}^m$, $m := \max(2\dim M + 1, n)$, and a diffeomorphism $\phi : M \longrightarrow M'$ such that:

$$i_{M'} \circ \phi \text{ is arbitrarily } \mathscr{C}^\infty_w\text{-close to } i_M .$$

Remarks: (i) If M is a Nash manifold $\Longrightarrow$ ϕ is Nash.

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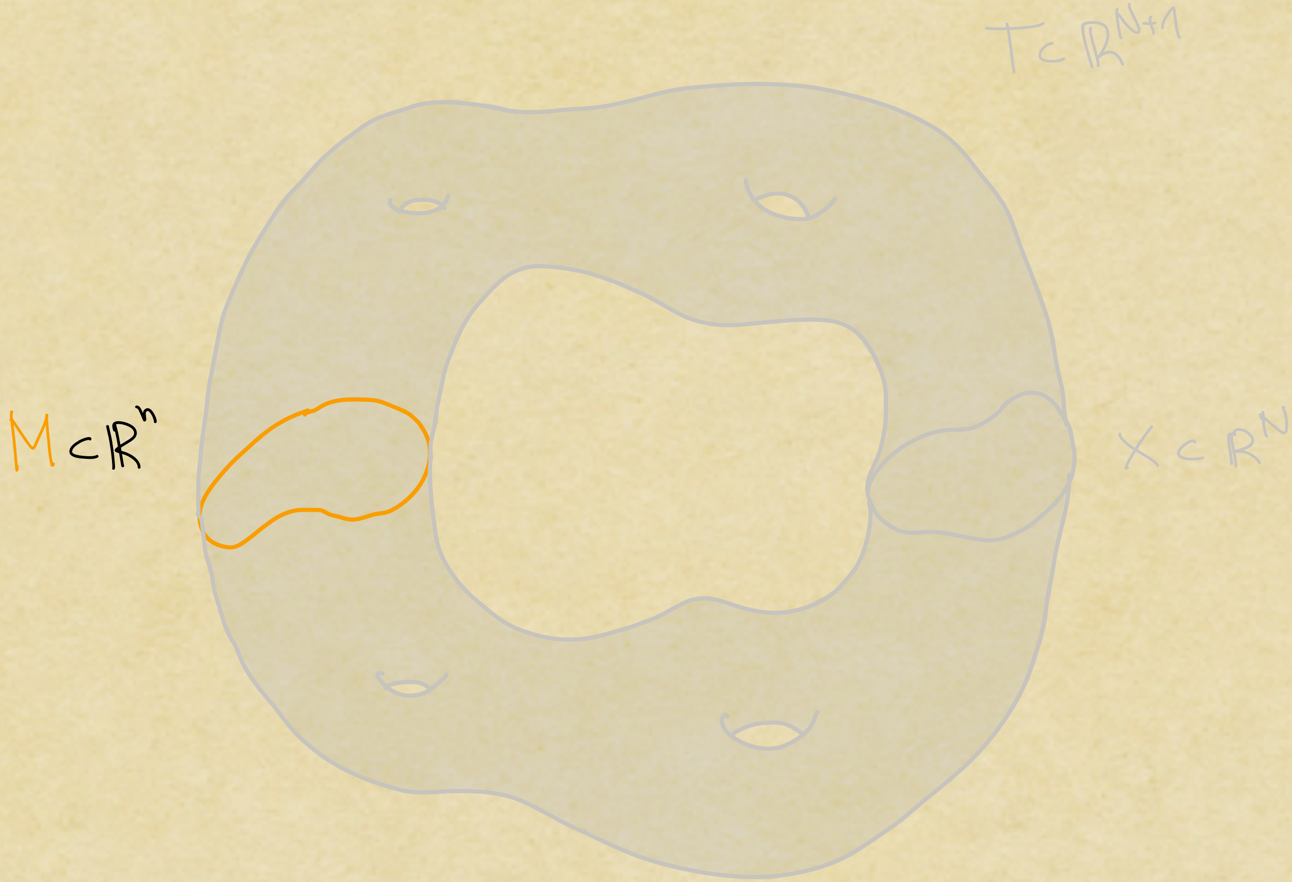

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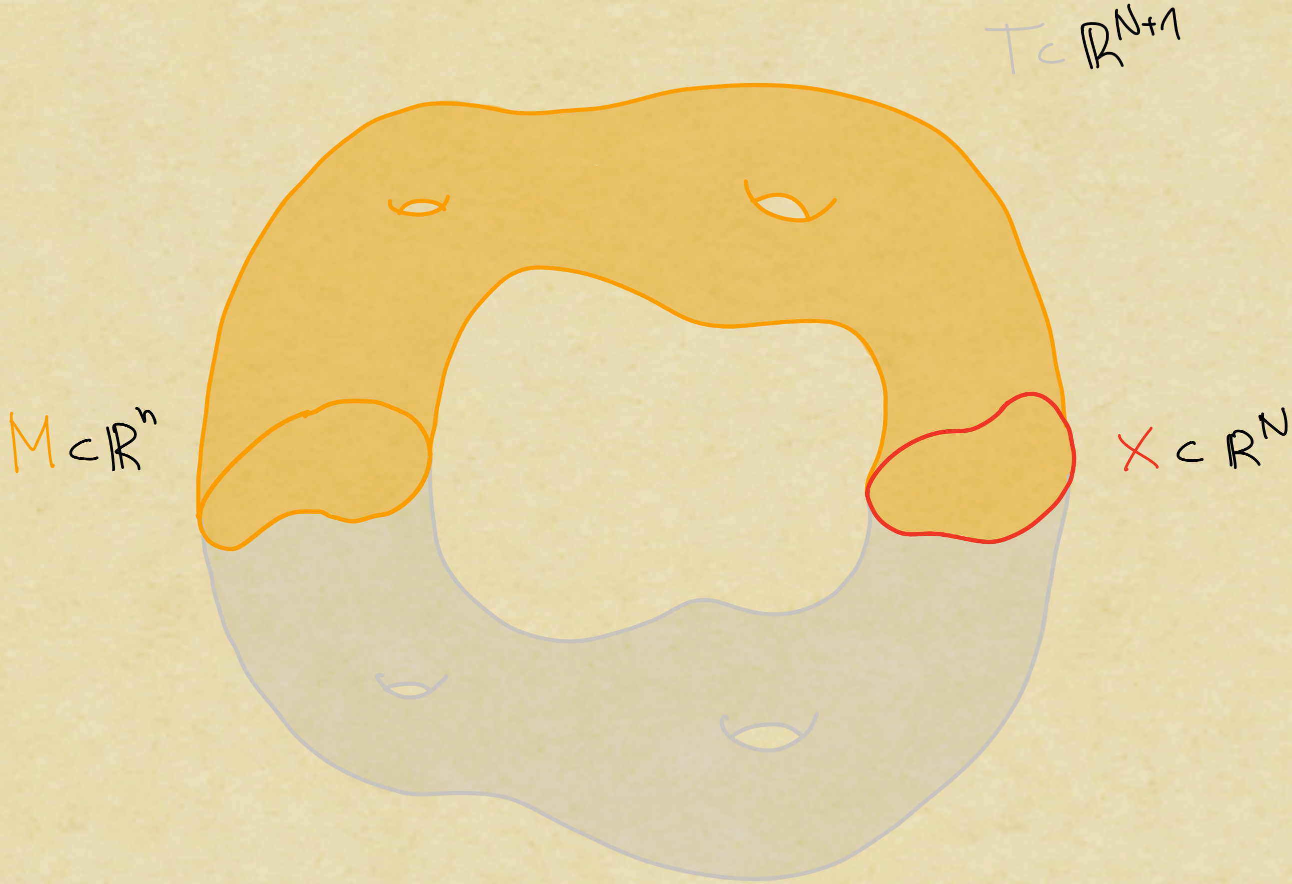

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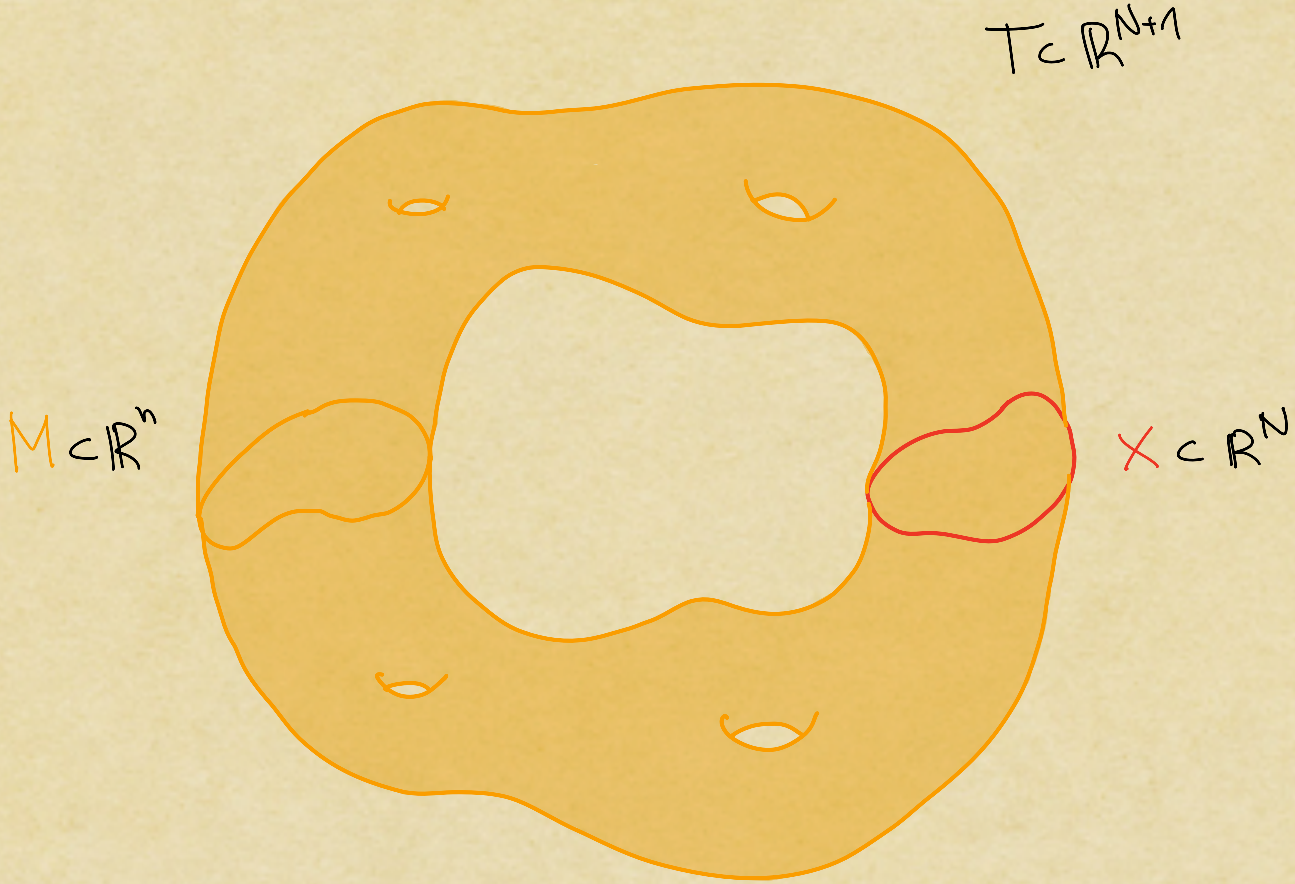

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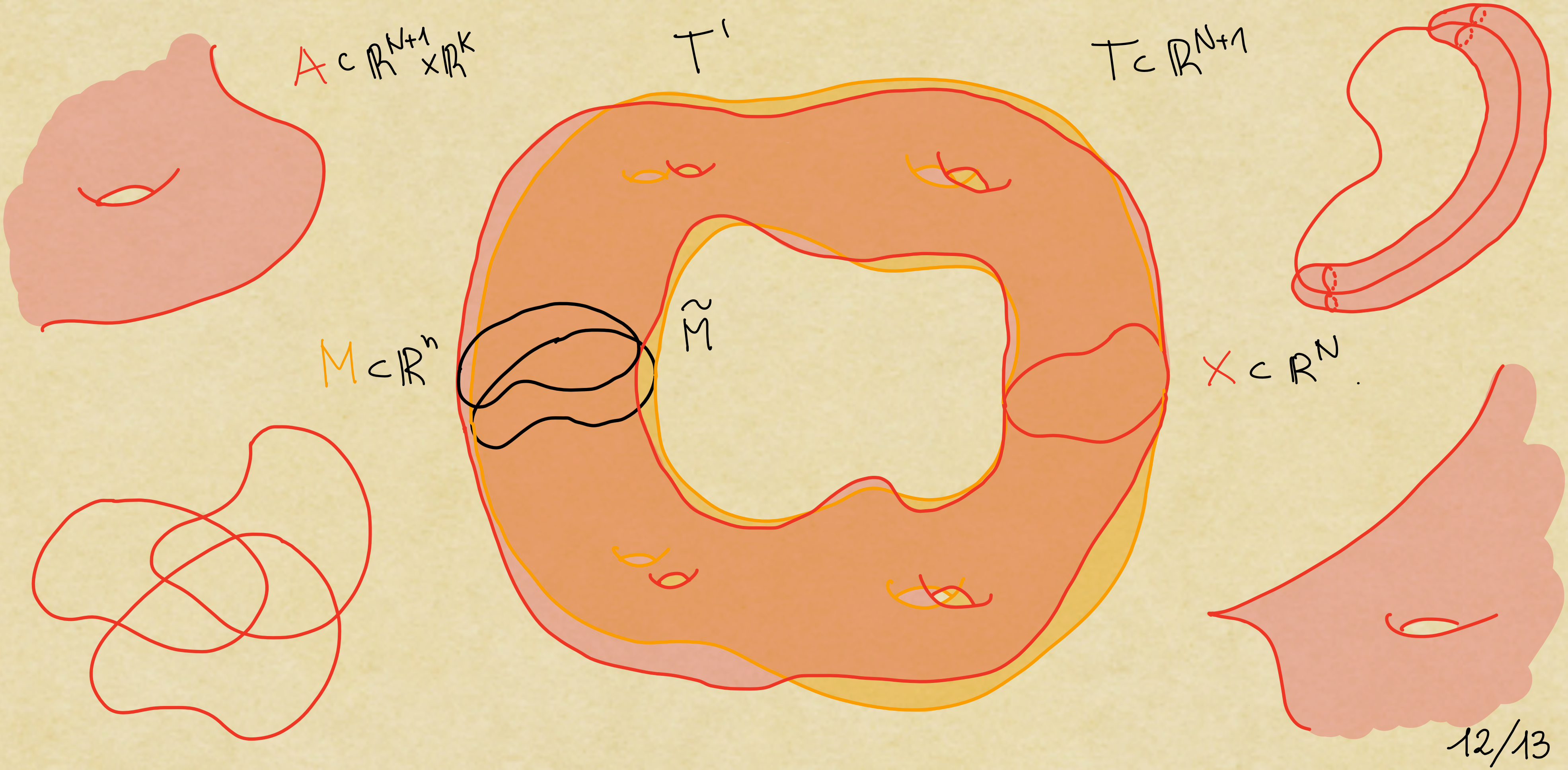

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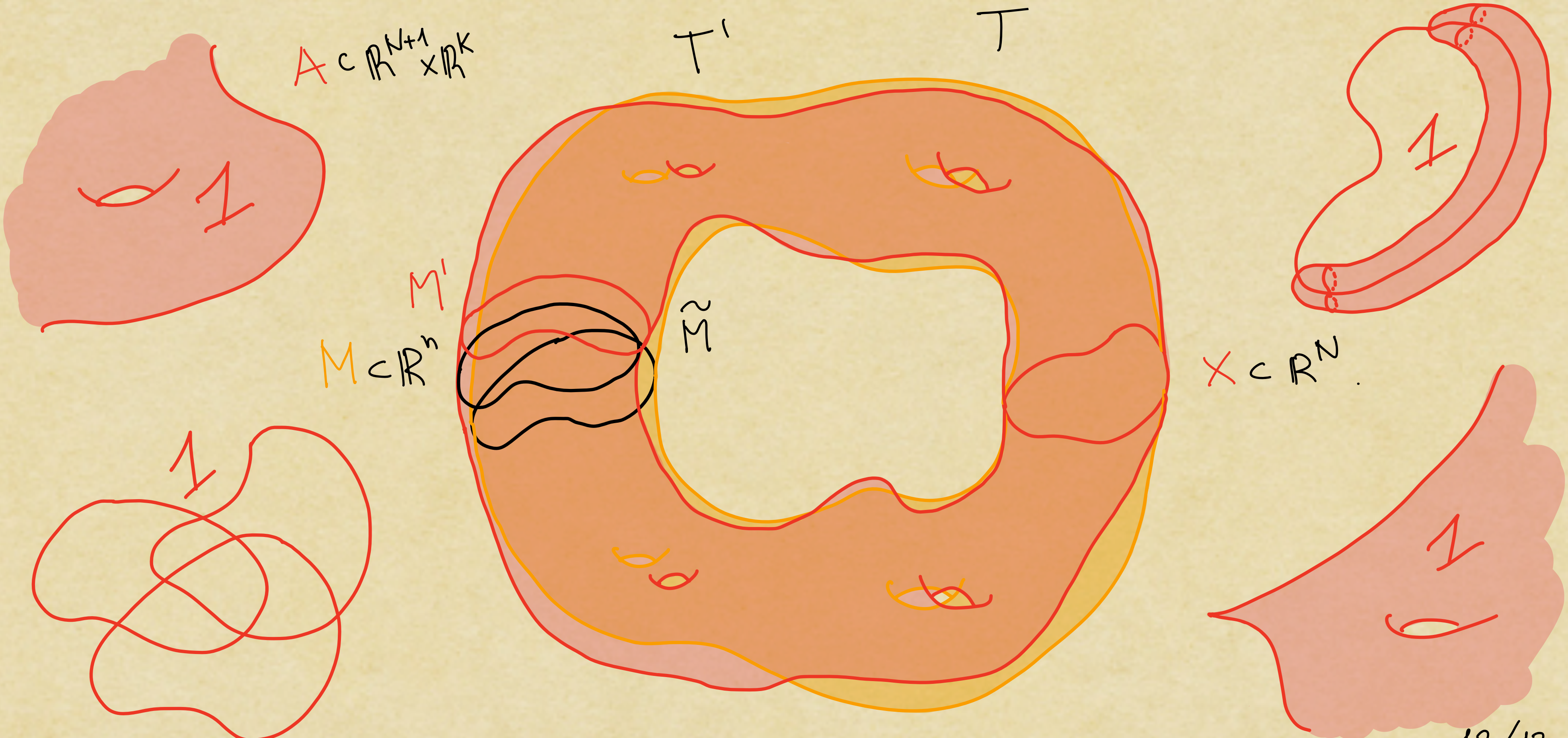

(III) SEPARATION IN $\mathbb{Q}$-IRREDUCIBLE $\mathbb{Q}$-ALGEBRAIC COMPONENTS

(5) M' is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.

Proposition: Let $X, Y \subset \mathbb{R}^n$ be $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of dimension $d < n$ such that $X \subsetneq Y$. Then, $Y \setminus X$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.

$M' \sqcup X$, X are $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic of dimension d $\Longrightarrow$ M' is so.

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Thanks for your attention !

References:

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