Tame geometry and extension of functions Pantucki 70

The Nash-Tognoli theorem over a

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Kraków, June 23, 2025

Q-algebraicity problem: (Parusiński, '21)

Is every algebraic set X c R" homeomorphic to some algebraic set X' c R" defined by polynomial equations with rational officients?

estimates depending on dim (X)

additional regularity + approximation

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3 Description over D'

Denote by $\overline{\mathbb{Q}}' := \overline{\mathbb{Q}} \cap \mathbb{R}$ the field of real algebroic mumbers.

Theorem: (Parusiński-Rond, 20) let XCR" be an algebraic set.

Then, there exists an algebraic set X'CR" such that:

- 1. There is an homeomorphism h: Rm -> Rn such that h(X) = X'.
- 2. h is semiologebraic and arc-analytic.
- 3. If $p_1, -, p_e \in RTx$) are such that $X = \frac{1}{2}p(p_1, -p_e)$, then $X' = \frac{1}{2}p(q_1, -, q_e)$ with $q_1, -q_e \in \overline{\mathbb{Q}}[x]$ and each q_i approximates p_i .

the proof men a Zariski-equisingular deformation of the coefficients of the polymonials $p_n(x)$, $p_e(x) \in R(x)$ s.t. $Z_{p_n}(p_n-p_e) = X$.

CRUCIAL PROPERTY: (Model completeness of the theory of real closed fields)

Let $S = \bigcup_{j=1}^{t} \int_{j=1}^{S} \{q_{ij}(x) \times_{ij} 0\} \in \mathbb{R}^{N}$ with $(x_{ij} \in \mathbb{R}^{T}) = \mathbb{R}^{T}$ be a semiologebraic set them: $S \cap (\mathbb{R}^{T}) = \emptyset$

Counterexample (Teissier '90, Parusinski-Panneson '25)

there is a surface singularity (X,0) c (43,0) whose equisingular

class does not admit any Delgebraic representative

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CRUCIAL PROPERTY: (Model completeness of the theory of real closed fields)

Let $S = \bigcup_{j=1}^{t} \bigcap_{j=1}^{S_{j}} \{q_{ij}(x) *_{ij} 0\} \subset \mathbb{R}^{n}$ with $*_{ij} \in \{1, 2\}$, $q_{ij} \in \mathbb{R}[x]$

be a semiolgebraic set them:

 $Sn(\overline{\mathbb{Q}}^r)^{\tilde{}} \neq \emptyset \iff S \neq \emptyset$

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CRUCIAZ PROPERTY: (Model completeness of the theory of real closed fields)

Let $S = \bigcup_{j=1}^{t} \bigcap_{j=1}^{s_{j}} \{q_{ij} \otimes x_{ij} \circ \} \subset \mathbb{R}^{n}$ with $\{x_{ij} \in \{1, 2\}, q_{ij} \in \mathbb{R}[x]\}$

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 $Sn(\overline{\mathbb{Q}}^r)^* \neq \emptyset \iff S \neq \emptyset$

Courterexample (Teissier '90, Parusiński-Paumeson '25)

there is a surface singularity $(X,0) \in (4^3,0)$ whose equisingular class does not admit any Q-algebraic representative.

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tons cone: Compact nousingulou algebraic hypernurfaces McR' ept nousingulor algebraic hypersuitace, chasse: (i) PERTX] s.t. Zpm(P)=M & \(\nabla_p(a) \neq 0\) \(\forall a \in M\)

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tony cone: Compact nousingular algebraic hypersurfaces

McR' est nousingulor algebraic hypersuitace, chare:

(1)
$$p \in R[X]$$
 s.t. $Z_{R^n}(p) = M$ & $\nabla p(a) \neq 0$ $\forall a \in M$

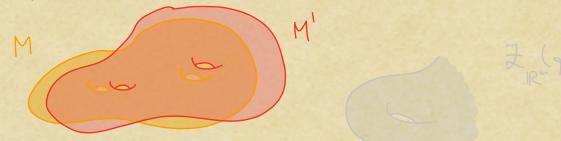
(i)
$$f \in C^{\infty}(\mathbb{R}^{n})$$
 , t. $f|_{U'} = p|_{U'}$ & $f|_{\mathbb{R}^{n}\setminus U} = 1$.

(ii) U'CUCIR^n a cpt meighborhood of Min R^n.

Uniform

Weicstrass approximation + Thom isotopy lemma

$$\Rightarrow$$
 there is gEQ[x] s.t. $M = Z$ [a) $Y \cong M$. $Z_{R^n}(g)$



& Q-regularity

Let $X \subset \mathbb{R}^m$ be a \mathbb{R} -olgebrouic set and $a = (a_1, -a_m) \in X$. Let: $n_a := (x_1 - a_1, -, x_m - a_m)$ and $\mathcal{I}_{\mathbb{Q}}(X) := \mathcal{I}(X) \cap \mathbb{Q}[X]$.

Definition: (Fermanolo-Ghiloni) We define the R-local ring $R_{x,a}^{Q} := R_{x,a} / T_{x,a} / T_{x,a} = R_{x,a}$.

We say that a \in Reg(X) is \mathbb{R} -regular if the ring $\mathbb{R}_{X,a}^{\mathbb{R}}$ is a regular local ring of dimension $\dim(X)$. We denote by $\operatorname{Reg}(X) \subset \operatorname{Reg}(X)$ the net of \mathbb{R} -regular points of X.

Remark: $\phi \neq \text{Reg}^{(X)} \subset \text{Reg}(X)$ is Isriski open & Sing (X)=X\Reg (X) is Q-algebraic.

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Definition: (Fermanolo-Ghilomi) We define the Q-local ring $R_{x,a}^{Q} := R(x)_{m_a} / T_{Q}(x) R(x)_{m_a}$. We say that $a \in Reg(x)$ is Q - regular if the ring $R_{x,a}^{Q}$

is a regular local ring of dimension $\dim(X)$. We denote by $\operatorname{Reg}(X) \subset \operatorname{Reg}(X)$ the net of \mathbb{Q} -regular points of X.

Remark: $\phi \neq \text{Reg}(x) \subset \text{Reg}(x)$ is Zariski open & Sing(x)=X\Reg(x) is Q-objectionic.

Definition: Let $X \subset \mathbb{R}^m$ be a \mathbb{R} -algebraic set. We say that $X \subset \mathbb{R}^m$ is \mathbb{R} -determined if $\operatorname{Reg}(X) = \operatorname{Reg}(X)$. \mathbb{R} -monsingular of \mathbb{R} - $\mathbb{R$

D- Jacobiou criterion: let X be a D-algebraic set. Then, and Reg(X) \iff there are $q_1, -, q_{m-d} \in L_D(X)$ such that $O(Q_1(a), -, V_{q_{m-d}(a)})$ are linearly independent over P(A) there is an Euclidean meighborhood O(A) of a in P(A) such that O(A) = O(A) and O(A) = O(A) and O(A) = O(A) and O(A) = O(A) are O(A) = O(A).

Définition: let XCIRM be a Q-algebraic set. We say that XCIRM is

- \mathbb{Q} -determined if $\operatorname{Reg}(X) = \operatorname{Reg}^{\mathbb{Q}}(X)$.
- Q-nonsingular if $X = Reg^{\mathbb{Q}}(X)$.

Q- Jacobiou criterion: let X be a Q-algebraic set. Theu,

at Reg(X) \iff there are $q_1, -, q_{n-d} \in I_{\mathbb{Q}}(X)$ such that

- 1 Tg,(a), _, Tg, d(a) are linearly independent over the
- (a) there is an Euclidean neighborhood U of a in R" such that

 $\times \cap \mathcal{O} = \mathcal{I}_{\mathbb{R}}(q_1, -, q_{m-d}) \cap \mathcal{O}$

Définition: let XCIRM be a Q-algebraic set. We say that XCIRM is

- · Reg(X) = Reg(X).
- Q-monsingular if $X = Reg^{Q}(X)$.

Q- Jacobiou criterion: let X be a R-algebraic set. Theu,

at Reg(X) \iff there are $q_1, -, q_{m-d} \in I_Q(X)$ such that

- (1) Vqn(a), _, Vqm-d(a) are linearly independent over the
- (ii) there is an Euclidean neighborhood U of a in R such that

 $\times \cap \cup = \mathcal{I}_{R}(q_1, -, q_{m-d}) \cap \cup$.

Lemma (Weierstrose approximation of the functions)

Let $X \subset \mathbb{R}^n$ be a compact Amousingular Adgebraic ret. Let $F \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ such that $X \subset \mathbb{R}_{\mathbb{R}^n}(F)$. Then, there is $q \in \mathbb{L}_{\mathbb{R}}(X)$ arbitrarily $\mathcal{C}^{\infty}_{W^{-}}$ close to F.

proof: Let $T_{Q}(X) = (q_1, \dots, q_C)$ and $a \in \mathbb{R}^m$. There is a meighborhood of a in \mathbb{R}^m and $u_1, \dots u_C \in \mathbb{C}(N_0)$ much that $f|_{U_{Q}} = \sum_{i=1}^{C} u_i \ q_i$ (1) If $a \in X$, by the Q-Jacobian criterion, there are $q_{i_1} \dots q_{i_{m-1}} \in \{q_{i_1}, \dots, q_C\}$ at

Lemma (Weierstross approximation of the functions)

Let $X \subset \mathbb{R}^n$ be a compact Amousingular Adgebraic set. Let $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ such that $X \subset \mathbb{R}^n(f)$. Then, there is $g \in \mathcal{I}(X)$ and that $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ such that $f \in \mathcal{C}^{\infty}(f)$.

proof: Let $T_{\mathbb{Q}}(X) = (q_1, -, q_{\mathbb{C}})$ and $a \in \mathbb{R}^m$. Here is a meighborhood

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(1) If a EX, by the D-Jacobian criterion, there are

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Lemma (Weierstross approximation of the functions) let $X \subset \mathbb{R}^n$ be a compact Amousingular Adgebraic ret. Let $F \in \mathcal{C}(\mathbb{R}^n)$ much that $X \subset \mathcal{I}_{\mathbb{R}^n}(F)$. Then, those is $q \in \mathcal{I}_{\mathbb{R}}(X)$ arbitrarily \mathcal{C}_{W}^{∞} -close to F.

proof: Let $T_Q(X) = (q_1, \dots, q_e)$ and $a \in \mathbb{R}^m$. Here is a meighborhood of a in \mathbb{R}^m and $u_1, \dots, u_e \in e^e(U_e)$ much that $f|_{U_a} = \sum_{i=1}^e u_i \cdot q_i \ .$

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Lemma (Weierstross approximation of the functions)

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(1) If $a \in X$, by the Q-Jacobian criterion, there are $q_{i_1}, -q_{i_{m-1}} \in \{q_{n-1} - q_{e}\}$ s.t.

(i) Tqi(a), _, Tqind (a) are linearly independent Na OX = Na O Z (gin - gind). (in) f = Z ni, gi, then approximate each vie EG(R") by pi ERIX) over K

(i) $\nabla q_{in}(a)$, _ , $\nabla q_{im-d}(a)$ are linearly independent (in) $Na \cap X = Na \cap Z_{R^n}(q_{in} - q_{ind})$. => there are a neighborhood Na sud Ni, -, Nin-d E Co (Na) n.t. $f = \sum_{j=1}^{\infty} u_{jj} q_{jj}$ then, just fix n; = 0 for every if in, _ i_m-ig. f = Z vi, gi, then approximate each viE (R) by pi ERIX) over K

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(i)
$$\nabla q_{in}(a)$$
, $-$, $\nabla q_{ind}(a)$ are timearly independent

(ii) $\mathcal{N}a \cap X = \mathcal{N}a \cap \mathbb{E}_{\mathbb{R}^n}(q_{in}, -q_{ind})$.

There are a neighborhood $\mathcal{N}a$ and \mathcal{N}_{ij} , $-$, $\mathcal{N}_{ind} \in \mathbb{C}^\infty(\mathcal{N}a)$ at

$$f = \sum_{j=1}^{m-d} \mathcal{N}_{ij} \cdot q_{ij}$$

Then, just fix $\mathcal{N}_{i} = 0$ for every $i \notin \mathcal{N}_{in} - i_{m-d}\mathcal{N}_{ind}$.

(2) If $a \notin X$, then there is $q_i \in \mathcal{N}_{q_i} - q_e \mathcal{N}$ at $q_i(a) \neq 0$

$$f = \underbrace{f}_{q_i} \cdot q_i \quad , \quad \text{so take} \quad \mathcal{N}_{i} = \underbrace{f}_{q_i} \cdot \mathcal{N}_{j} = 0 \quad \forall j \neq i.$$

and $\mathcal{N}_{a} = \mathbb{R}^n / \underbrace{f}_{\mathbb{R}^m}(q_i)$.

Patch together the local descriptions by a partition of unity argument getting

 $f = \sum_{i=1}^{n} u_i \cdot q_i$, then approximate each $u_i \in \mathcal{C}(\mathbb{R}^n)$ by $p_i \in \mathbb{R}[x]$ over K.

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$$Na \cap X = Na \cap Z_{pr}(q_{in}, -q_{ind})$$
.

=> there are a neighborhood Na and Ni, -, Nin-d & Co (Na) n.t.

$$f = \sum_{j=1}^{n-d} x_{ij} \cdot q_{ij}$$

then, just fix $N_i = 0$ for every $i \notin \{i_n = i_{m-s}\}$.

(2) If at X, then there is qi E iqn, -qe's n.t. qi(a) # 0

$$= 7 \qquad f = \frac{f}{q_i} \cdot q_i \quad , \quad \infty \text{ take} \qquad n_i = \frac{f}{q_i} \quad , \quad n_j = 0 \quad \forall j \neq i.$$
 and $u_n = \mathbb{R}^n \setminus \pm_{\mathbb{R}^n}(q_i)$.

Patch together the local descriptions by a partition of unity argument getting $f = \sum_{i=1}^{\ell} u_i \cdot q_i$, then approximate each $u_i \in \mathcal{C}^0(\mathbb{R}^n)$ by $p_i \in \mathbb{R}[X]$ are K.

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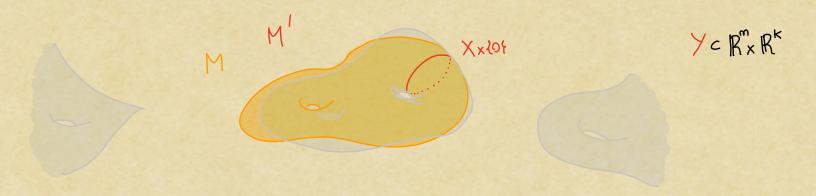
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Patch together the local descriptions by a partition of unity argument getting $f = \sum_{i=1}^{\ell} u_i \cdot q_i$, then approximate each $u_i \in \mathcal{C}^0(\mathbb{R}^n)$ by $p_i \in \mathbb{R}[X]$ are K.

Nach theorem over Q: (Ghiloni, S.) Let MCR" be a compact & monifold and let XCM be a Romonismental Redebraic xit.

Then, M is diffeomorphic fixing X to the union of some commedted components M' of a Redebraic xit YCR" xR' much that:

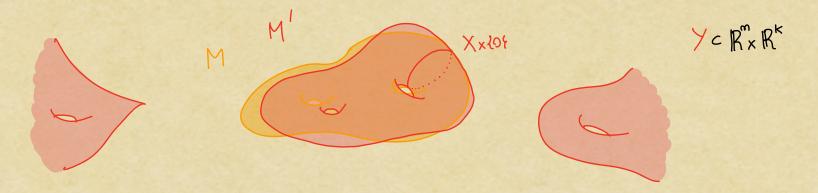
Xx101 CM' & M' C Reg (Y).



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Vach-Tagmoli theorem over \mathbb{R} : (Ghilomi-S.)

Let $M \in \mathbb{R}^m$ be a compact $\mathcal{C}=$ mountiald. There exists a \mathbb{R} -mousingular \mathbb{R} -depends on \mathbb{R} of \mathbb{R} or \mathbb{R} of \mathbb{R} of \mathbb{R} or \mathbb{R} of \mathbb{R} or \mathbb{R}

Remarks: in If M is a Nash monitold => \$\Phi\$ is Nash

(ii) ϕ can be obtained by an isotopy $\overline{\Phi}: M \times [0,1] \longrightarrow \mathbb{R}^M$ such that $\overline{\Phi} = 2$. 2 $\overline{\Phi} = 0$

Vash-Tagnoh theorem over Q: (Ghilomi-S.)

Let MCRM be a compact & moutold. There exists a Rmousingular Q digebroic out M' C Rm, m:=max (2 dim M+1, m), and a diffeomorphism $\phi: M \to M'$ such that:

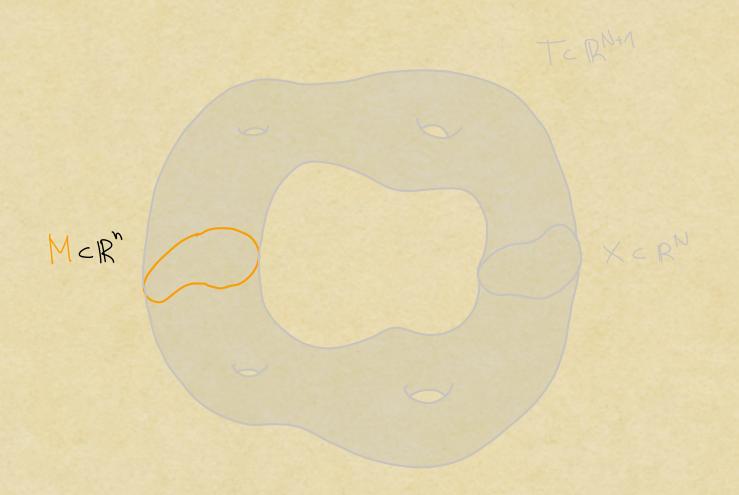
in of is arbitrarily endone to in.

Remarks: i) If M is a Nash monifold => \$\Phi\$ is Nash.

(iii) ϕ can be obtained by an isotopy $\overline{\Phi}: M \times [0,1] \longrightarrow \mathbb{R}^m$ such that $\overline{\Phi}_0 = z_M$ & $\overline{\Phi}_1 = \phi$

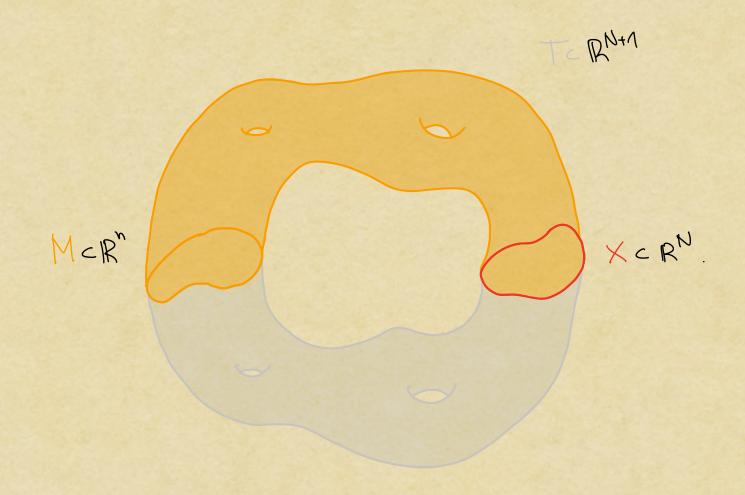
(I) PREPARATION

- (1) D'monsingular Dalgebraic representatives of the abordism group
- (2) Doubling the embedded cobordism



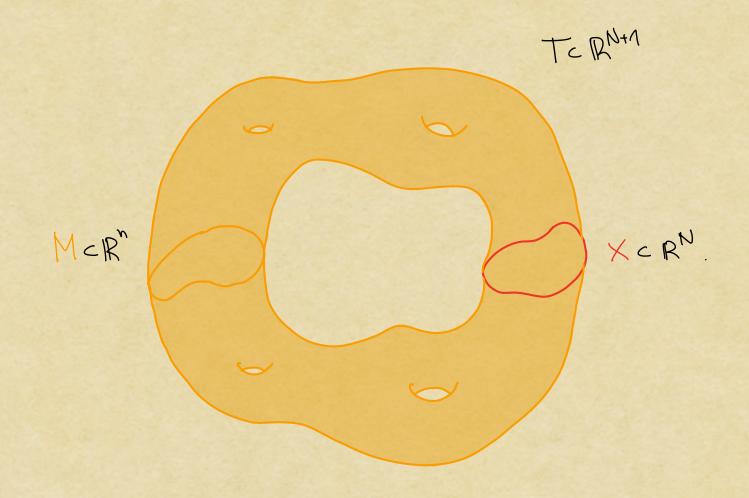
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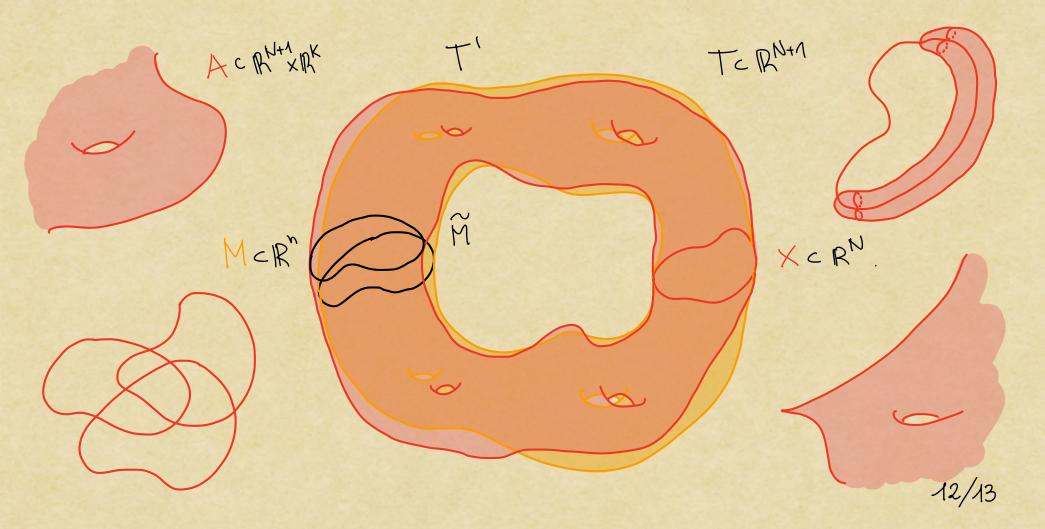
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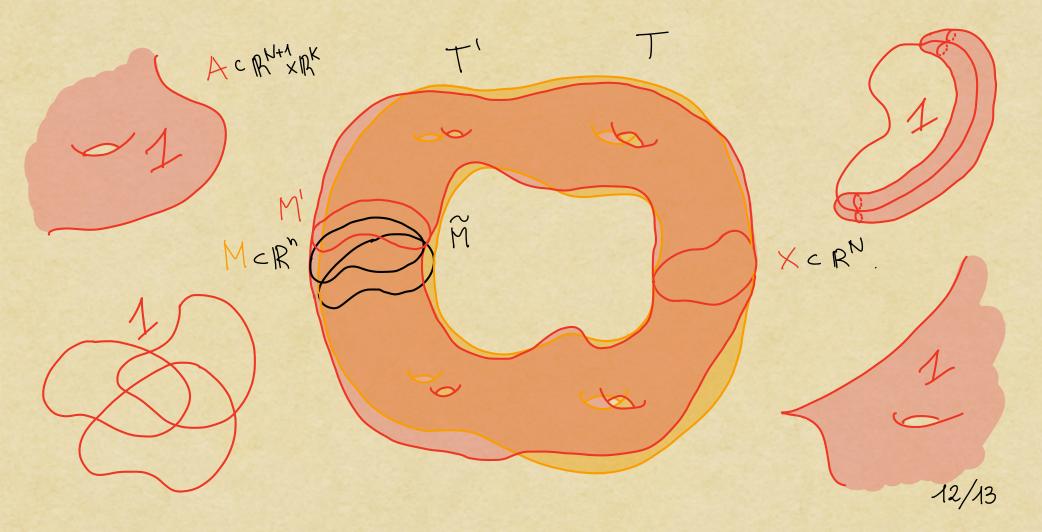
(II) Q-APPROXIMATION

- (3) Nash theorem over Q fixing X
- (4) R-approximation of MUX in T



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- (3) Nash theorem over Q fixing X
- (4) R-approximation of MUX in T1.



(III) SEPARATION IN Q-IRREDUCIBLE Q-ALGEBRAIC COMPONENTS

(5) M' is a R-monsingular D-olgebraic set.

Proposition: Let X, Y c R" be Q mousingular Q-algebraic sets of dimension of 2 m such that X & Y. Then, Y X is a Q-monosingular Q-algebraic set.



RN+1 x RK



M'UX, X are Q-mousingulor Q-objetionic of dimension of => M' is so.

(6) Q-algebraic generic projection to have M'CRM, m=max(n,2d+1)

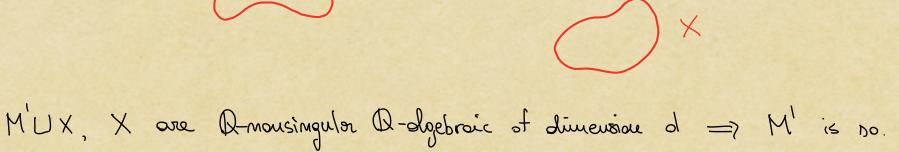
(III) SEPARATION IN Q-IRREDUCIBLE Q-ALGEBRAIC COMPONENTS

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(6) Q-algebraic generic projection to have $M' \subset \mathbb{R}^{m}$, $m = \max(m, 2d+1)$.

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thomks for your attention!

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