$\mathcal{C}^m$ EXTENSIONS OF SEMIALGEBRAIC FUNCTIONS

Joint work with E. Bierstone and P.D. Milman

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Recall the two problems discussed in Charles Fefferman's talk.

Whitney's extension problem

Let $X \subset \mathbb{R}^n$ be closed and $f : X \to \mathbb{R}$.
How to determine whether there exists a $\mathcal{C}^m$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F_{|X} = f$?

The Brenner–Epstein–Fefferman–Hochster–Kollár problem

Let $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ and $f : \mathbb{R}^n \to \mathbb{R}^p$.
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Question

If the data are semialgebraic, can we expect to obtain a semialgebraic solution ?

For instance, it is true for Whitney's extension theorem.

Theorem – Whitney 1934

Given a $\mathcal{C}^m$ Whitney field on a closed subset $X \subset \mathbb{R}^n$,
i.e. a family $(f_\alpha : X \to \mathbb{R})_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq m}}$ of continuous functions such that

$$\forall z \in X, \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m \implies f_\alpha(x) - \sum_{|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta = \underset{X\ni x,y\to z}{o}\left(\|x-y\|^{m-|\alpha|}\right),$$

there exists a $\mathcal{C}^m$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $D^\alpha F_{|X} = f_\alpha$ and F is analytic on $\mathbb{R}^n \setminus X$.

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Theorem – Kurdyka–Pawłucki, 1997, 2014
Kocel-Cynk–Pawłucki–Valette, 2019

Given a **semialgebraic** C^m Whitney field on a closed subset $X \subset \mathbb{R}^n$,
i.e. a family $(f_\alpha : X \to \mathbb{R})_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le m}}$ of continuous **semialgebraic** functions such that

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The semialgebraic Whitney extension problem

Let $f : X \to \mathbb{R}$ be a semialgebraic function where $X \subset \mathbb{R}^n$ is closed.
If f admits a $\mathcal{C}^m$ extension $F : \mathbb{R}^n \to \mathbb{R}$, does it admit a semialgebraic $\mathcal{C}^m$ extension ?

The semialgebraic Brenner–Epstein–Fefferman–Hochster–Kollár problem

Let $A : \mathbb{R}^n \to \mathcal{M}_{p,q}(\mathbb{R})$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ be semialgebraic.
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Some positive results:

- Aschenbrenner–Thamrongthanyalak (2019): for $m = 1$ (for Glaeser bundles).
- Fefferman–Luli (2022): for $n = 2$ (for Glaeser bundles).
- Bierstone–C.–Milman (2021): $\forall n$, $\forall m$, with loss of differentiability.
- Parusiński–Rainer (2024): for the $C^{1,\omega}$ extension problem.

The semialg $\mathcal{C}^1$ extension problem (Aschenbrenner–Thamrongthanyalak, 2019)

Let $f : X \to \mathbb{R}$ be a semialgebraic function, $X \subset \mathbb{R}^n$ closed, admitting a $\mathcal{C}^1$ extension $F : \mathbb{R}^n \to \mathbb{R}$.

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- Set $S := \left\{ (x,y,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ : \ \begin{array}{l} x \in X,\ y = f(x), \\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall a,b \in B_\delta(x),\ |f(b) - f(a) - v \cdot (b-a)| \le \varepsilon \|b-a\| \end{array} \right\}$.

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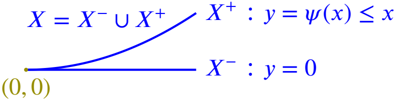
This strategy does not generalise when $m > 1$ since the unknown $(f_\alpha)_{\substack{\alpha \in \mathbb{N}^n \setminus \{\mathbf{0}\} \\ |\alpha| \leq m}}$ can't be described as a section of a set written using a first order formula. For instance, if $m = 2$, we must have

$$f_{\mathbf{e}_i}(b) = f_{\mathbf{e}_i}(a) + \sum_{j=1}^{n} f_{\mathbf{e}_i + \mathbf{e}_j}(a)(b_j - a_j) + \underset{X \ni a,b \to c}{o}(\|b - a\|).$$

The planar semialgebraic extension problem (Fefferman–Luli, 2021)

Let $f : X \to \mathbb{R}$ be semialgebraic where X is as on the right.
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^m function such that $F_{|X} = f$ and $J_{(0,0)}F = 0$.



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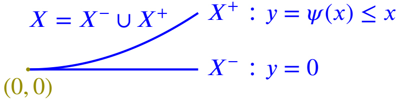

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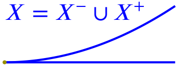

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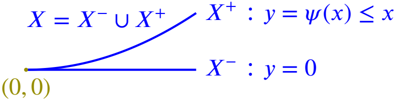

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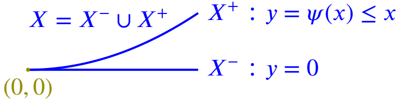

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3. Then $\tilde{F}(x,y) = \theta^-(x,y)\left(\sum_{l=0}^{m} \frac{\tilde{f}_l^-(x)}{l!} y^l\right) + \theta^+(x,y)\left(\sum_{l=0}^{m} \frac{\tilde{f}_l^+(x)}{l!}(y-\psi(x))^l\right)$ is a semialgebraic C^m extension of f in a neighbourhood of the origin such that $J_{(0,0)}\tilde{F} = 0$. ∎

For all n and m, with loss of differentiability

Theorem – Bierstone–C.–Milman, 2021

Given $X \subset \mathbb{R}^n$ closed and semialgebraic, there exists $r : \mathbb{N} \to \mathbb{N}$ satisfying the following property: if $f : X \to \mathbb{R}$ semialgebraic admits a $C^{r(m)}$ extension, then it admits a semialgebraic C^m extension.

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The results hold for functions *definable in an expansion of* $\mathbb{R}$ *by restricted quasianalytic functions* and not merely *semialgebraic*.

First step: towards a common generalisation (no Glaeser bundle involved)

The extension problem

Let $X \subset \mathbb{R}^n$ be semialgebraic and closed.

There exists $\varphi : M \to \mathbb{R}^n$ Nash and proper such that $X = \varphi(M)$.

Given $g : \mathbb{R}^n \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have

$$g_{|X} = f \text{ if and only if } \forall y \in M,\ g(\varphi(y)) = \tilde{f}(y)$$

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The equation problem

Consider the equation

$$A(x)G(x) = F(x),\ \ x \in \mathbb{R}^n.$$

with semialgebraic data.
After composing with $\varphi : M \to \mathbb{R}^n$ some Nash and proper map, we reduce to

$$\tilde{A}(y)G(\varphi(y)) = \tilde{F}(y),\ \ y \in M$$

where $\tilde{A} := A \circ \varphi$ is now Nash and $\tilde{F} := F \circ \varphi$.

Therefore, it is enough to solve

$$A(x)g(\varphi(x)) = f(x)$$

where

- $A : M \to \mathcal{M}_{p,q}(\mathbb{R})$ is Nash,
- $\varphi : M \to \mathbb{R}^n$ is Nash and proper, and,
- the unknown is $g : \mathbb{R}^n \to \mathbb{R}^q$.

Goal: if there is a $\mathcal{C}^{r(m)}$ solution then there is a semialgebraic $\mathcal{C}^m$ solution.

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Actually, we are looking for a formal solution of

$$\forall b \in \varphi(M),\ \forall a \in \varphi^{-1}(b),\ T_a^m f(\mathbf{x}) \equiv T_a^m A(\mathbf{x})\, G(b, T_a^m \varphi(\mathbf{x})) \mod (\mathbf{x})^{m+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

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where the unknown is a semialgebraic $\mathcal{C}^m$ Whitney field

$$G(b,\mathbf{y}) = \sum_{\substack{|\alpha|\le m \\ j=1,\dots,q}} \frac{g_{\alpha,j}(b)}{\alpha!}\mathbf{y}^{\alpha,j} \in \left(\mathcal{C}^0(X)[\mathbf{y}]\right)^q$$

Therefore, it is enough to solve

$$A(x)g(\varphi(x)) = f(x)$$

where

- $A : M \to \mathcal{M}_{p,q}(\mathbb{R})$ is Nash,
- $\varphi : M \to \mathbb{R}^n$ is Nash and proper, and,
- the unknown is $g : \mathbb{R}^n \to \mathbb{R}^q$.

Goal: if there is a $\mathcal{C}^{r(m)}$ solution then there is a semialgebraic $\mathcal{C}^m$ solution.

Actually, we are looking for a formal solution of

$$\forall b \in \varphi(M),\ \forall a \in \varphi^{-1}(b),\ T_a^m f(\mathbf{x}) \equiv T_a^m A(\mathbf{x})\, G(b, T_a^m \varphi(\mathbf{x})) \mod (\mathbf{x})^{m+1}\mathbb{R}[\![\mathbf{x}]\!]^p$$

where the unknown is a semialgebraic $\mathcal{C}^m$ Whitney field

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where
$\mathbf{y}^\alpha = y_1^{\alpha_1}\cdots y_n^{\alpha_n}$
$\mathbf{y}^{(\alpha,j)} = (0,\dots,0,\mathbf{y}^\alpha,0,\dots,0)$

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$$\forall b \in B, \forall a \in \varphi^{-1}(b),\ T_a^l f(\mathbf{x}) \equiv T_a^l A(\mathbf{x})\, G(b, T_a^l \varphi(\mathbf{x})) \mod (\mathbf{x})^{l+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{E}$$

Proposition: induction on the dimension

Let $B \subset \varphi(M)$ be a semialgebraic closed subset.
There exists $B' \subset B$ semialgebraic and closed with $\dim(B') < \dim(B)$ such that if (E) admits a semialgebraic solution on B' then it admits a solution on B modulo a loss of differentiability.

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Strategy by successive flattening:

1. Construct a suitable B';
2. Assume that $T_a^l f(\mathbf{x}) \equiv 0$ on B';
3. Find a semialg solution $G(b, \mathbf{y}) \in \left(C^0(X)[\mathbf{y}]\right)^q$ s.t. $G_{|B'} \equiv 0$ and $G_{|B \setminus B'}$ is a C^l Whitney field.

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A - Whitney regularity

Given B, there exists $\rho \in \mathbb{N}$ such that if G is a Whitney field of order $l \geq k\rho$ on $B \setminus B'$ then it is a Whitney field of order k on B.

$$\forall b \in B,\ \forall a \in \varphi^{-1}(b),\ T_a^r f(\mathbf{x}) \equiv T_a^r A(\mathbf{x})\, G(b, T_a^r \varphi(\mathbf{x})) \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{E}$$

B - Chevalley's function

For $l \in \mathbb{N}$, there exists $r \geq l$ such that the coefficients $g_{\alpha,j}$, $|\alpha| \leq l$, are entirely determined by (E).

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$$\forall b \in \Lambda_\tau,\ \pi_l(\mathcal{R}_r(b)) = \pi_l(\mathcal{R}_{r-1}(b))$$

where

- $\mathcal{R}_r(b)$ is the *module of relations at* b

$$\mathcal{R}_r(b) := \left\{ W \in \mathbb{R}[\![\mathbf{y}]\!]^q \ :\ \forall a \in \varphi^{-1}(b),\ T_a^r A(\mathbf{x}) W\left(\tilde{T}_a^r \varphi(\mathbf{x})\right) \equiv 0 \mod (\mathbf{x})^{r+1}\mathbb{R}[\![\mathbf{x}]\!]^p \right\}$$

- π_l is the truncation up to degree l.

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Set $B' := \bigsqcup_{\dim \Lambda_\tau < \dim B} \Lambda_\tau$.

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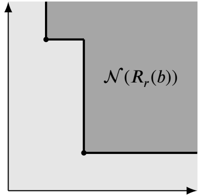
We totally order $\mathbb{N}^n \times \{1, \dots, q\} \ni (\alpha, j)$ by $\operatorname{lex}(|\alpha|, j, \alpha_1, \dots, \alpha_n)$.
Then the *diagram of initial exponents* of $\mathcal{R}_r(b)$,

$$\mathcal{N}(\mathcal{R}_r(b)) := \left\{ \exp W \ : \ W \in \mathcal{R}_r(b) \setminus \{0\} \right\} \subset \mathbb{N}^n \times \{1, \dots, q\},$$

is constant on Λ_τ (up to refining).



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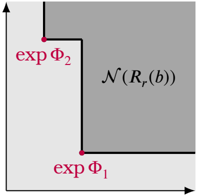

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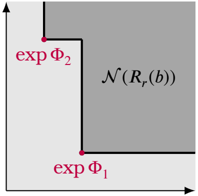

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By the existence of a C^t solution[1], there exists $W_b \in \mathbb{R}[\mathbf{y}]^q$ such that

$$T_a^t f(\mathbf{x}) \equiv T_a^t A(\mathbf{x})\, W_b(T_a^t \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p, \quad \forall a \in \varphi^{-1}(b).$$

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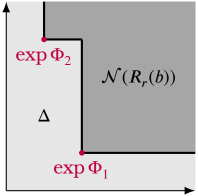

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By Hironaka's formal division, we may write

$$W_b(\mathbf{y}) = \sum_{i=1}^{s} Q_i(\mathbf{y})\Phi_i(\mathbf{y}) + V_\tau(b, \mathbf{y})$$

where $Q_i \in \mathbb{R}[\![\mathbf{y}]\!]$, $V_\tau \in \mathbb{R}[\![\mathbf{y}]\!]^q$ and $\operatorname{supp} V_\tau(b, \mathbf{y}) \subset \Delta := \mathcal{N}(\mathcal{R}_r(b))^c$.

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$$\forall b \in B,\ \forall a \in \varphi^{-1}(b),\ T^t_a f(\mathbf{x}) \equiv T^t_a A(\mathbf{x})\, G(b, T^t_a \varphi(\mathbf{x})) \mod (\mathbf{x})^{t+1}\mathbb{R}[\![\mathbf{x}]\!]^p \tag{E}$$

Claim

$G_\tau(b, \mathbf{y}) := \pi_l\left(V_\tau(b, \mathbf{y})\right)$ is a C^l semialgebraic Whitney field on Λ_τ satisfying (E).

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Proof.

- G_τ is semialgebraic since it may be expressed in terms of the data.
- $G_\tau(b, \mathbf{y})$ satisfies (E) since it differs from $W_b(\mathbf{y})$ by an element of $\mathcal{R}_r(b)$.
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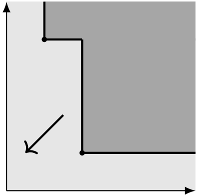

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C - gluing between strata using Łojasiewicz inequality

There exists $\sigma \in \mathbb{N}$ such that if $t \geq r + \sigma$ then $\lim_{b \to \overline{\Lambda_\tau} \setminus \Lambda_\tau} G_\tau(b, \mathbf{y}) = 0$.

Note that $\overline{\Lambda_\tau} \setminus \Lambda_\tau \subset B'$.

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Summary: loss of differentiability

For $k \in \mathbb{N}$, we set $l \geq k\rho$, then $r \geq l$ and finally $t \geq r + \sigma$ where

A. ρ is an upper bound of Whitney's loss of differentiability (induction step).
B. r is an upper bound of the Chevalley functions on the various strata.
C. σ is an upper bound of Łojasiewicz's loss of differentiability on each stratum.

Conclusion.
Assuming the existence of a C^t solution, we constructed a semialgebraic solution of (E)

$$G(b, \mathbf{y}) \in \left(C^0(B)[\mathbf{y}]\right)^q$$

such that G is a C^l Whitney field on $B \setminus B'$ and $G_{|B'} = 0$.
Therefore G is a semialgebraic C^k Whitney field on B.